Contrasting two approaches in real options valuation: contingent claims versus dynamic programming

M.C. Insley T.S. Wirjanto [1] [2]

July 2, 2009

[1]Margaret Insley is with the Department of Economics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Phone: 519-888-4567 ext 32422, Email: minsley@uwaterloo.ca. Tony Wirjanto is with the School of Accounting & Finance and the Department of Statistics & Actuarial Science, University of Waterloo, Waterloo, Ontario, N2L 3G1; Phone: 519-888-4567 ext. 35210; Email: twirjant@uwaterloo.ca

[2]The authors thank Ken Vetzal, Graham Davis, Robert Cairns, Patrick Martin, and Peter Forsyth for their comments. Any errors are of course the responsibility of the authors. Thanks also to the Social Sciences and Humanities Research Council of Canada for financial support

Abstract

Contrasting two approaches in real options valuation: contingent claims versus dynamic programming

This paper compares two well-known approaches for valuing a risky investment using real options theory: contingent claims (CC) with risk neutral valuation and dynamic programming (DP) using a constant risk adjusted discount rate. Both approaches have been used in valuing forest assets. A proof is presented which shows that, except under certain restrictive assumptions, DP using a constant discount rate and CC will not yield the same answers for investment value. A few special cases are considered for which CC and DP with a constant discount rate are consistent with each other. An optimal tree harvesting example is presented to illustrate that the values obtained using the two approaches can differ when we depart from these special cases to a more realistic scenario. We conclude that for real options problems the CC approach is preferred when data exists (such as futures prices) that allow the estimation of the market price of risk or convenience yield. Even when such data do not exist we argue that the CC approach is preferred as it has the advantage of allowing the individual specification of the prices of different sources of risk.

Keywords: optimal tree harvesting, real options, contingent claims, dynamic programming

Running Title: Contingent claims theory and dynamic programming

1 Introduction

Over the past two decades, developments in the theory and methodology of financial economics have been applied to advantage to general problems of investment under uncertainty. The well known book by Dixit and Pindyck [1994] draws the analogy between valuing financial options and investments in real assets or real options which involve irreversible expenditures and uncertain future payoffs depending on one or more stochastic underlying variables. Natural resource investments, including forestry, provide a good application of real options theory as their value depends on volatile commodity prices and they entail decisions about the timing of large irreversible expenditures.[1]

Two particular approaches used in the real options literature are dynamic programming (DP) and contingent claims (CC). DP is an older approach developed by Bellman and others in the 1950's and used extensively in management science. DP involves formulating the investment problem in terms of a Hamilton-Jacobi-Bellman (HJB) equation and solving for the value of the asset by backward induction using a discount rate which reflects the opportunity cost of capital for investments of similar risk. In practice dynamic programming typically involves adopting an exogenous constant discount rate.

The contingent claims approach has its origins in the seminal papers of Black and Scholes [1973] and [Merton, 1971, 1973] and is now standard in many finance texts.[2] This approach assumes the existence of a sufficiently rich set of markets in risky assets so that the stochastic component of the risky project under consideration can be exactly replicated. Through appropriate long and short positions, a riskless portfolio can be constructed consisting of the risky project and investment assets which track the project's uncertainty. In equilibrium

[1]Examples of applications of real options theory to natural resources include Paddock et al. [1988], Brennan and Schwartz [1985], Schwartz [1997], Slade [2001] and references therein, Harchaoui and Lasserre [2001], Mackie-Mason [1990], Saphores [2000], and papers contained in Schwartz and Trigeorgis [2001]. A review of the empirical significance of real options in valuing mineral assets is contained in Davis [1996].

[2]See Hull [2006] and Ingersoll [1987] for example. Dixit and Pindyck [1994] and Trigeorgis [1996] present contingent claims in a real options context.

with no arbitrage opportunities, this portfolio must earn the risk free rate of interest, which allows the value of the risky project to be determined. The no-arbitrage assumption avoids the necessity of determining the appropriate risk adjusted discount rate. However if a portion of the return from holding the risky asset is due to an unobservable convenience yield, it is still necessary to estimate either that convenience yield or a market price of risk, which is often problematic.[3]

Both CC and DP have been used in the natural resources literature. For example Slade [2001] and Harchaoui and Lasserre [2001] use a contingent claims approach to value mining investments. In the forestry economics literature, the DP approach has generally dominated. An exception is Morck et al. [1989] who use a CC approach along with an assumed convenience yield for an application in forestry. In those forestry papers that use a DP approach, there is rarely much discussion of the choice of discount rate. Sometimes a risk neutral setting is explicitly assumed allowing use of a riskfree discount rate; other times a rate is adopted without explanation. A selection of papers that use the dynamic programming approach include Clarke and Reed [1989], Haight and Holmes [1991], Thomson [1992], Yin and Newman [1997], Plantinga [1998], Gong [1999], Insley [2002], and Insley and Rollins [2005]. Alvarez and Koskela [2006] and Alvarez and Koskela [2007] deal with risk aversion by explicitly modelling the decision maker's subjective utility function.

One reason for the dominance of the DP approach in the forestry literature is likely due to the difficulty in estimating the convenience yield. In theory futures prices could be used to obtain such an estimate. Futures markets do exist for lumber, however currently the maturity of futures contracts is less than one year, while the typical optimal harvesting problem is applied to a very long lived investment, with stands of trees maturing over 40 to 70 years.[4]

[3]Dixit and Pindyck [1994] discusses the convenience yield in detail. It represents a return that accrues to the holder of the physical asset but not the holder of an option on the asset. For commodities such as copper, oil, or lumber the convenience yield represents the benefits of holding inventory rather than having to purchase the commodity in the spot market.

[4]There have been papers addressing this issue for other commodities including Gibson and Schwartz

In comparing the two approaches, Dixit and Pindyck [1994] note that each has advantages and disadvantages, but that CC provides a better treatment of risk. They point out that one problem with the investment rule derived from DP is that

> "(...)it is based on an arbitrary and constant discount rate, ρ. It is not clear where this discount rate should come from, or even that it should be constant over time" (page 147).

On the other hand, a disadvantage of CC is that it requires a sufficiently rich set of risky assets so that the risky components of the uncertain investment can be exactly replicated. This is not required in dynamic programming;

> "(...) if risk cannot be traded in markets, the objective function can simply reflect the decision maker's subjective valuation of risk. The objective function is usually assumed to have the form of the present value of a flow 'utility' function calculated using a constant discount rate, ρ. This is restrictive in its own way, but it too can be generalized. Of course we have no objective or observable knowledge of private preferences, so testing the theory can be harder" (page 121).

In a review of Dixit's and Pindyck's book, Schwartz [1994] disagrees with the common practice of using a discount rate that simply reflects the decision-maker's subjective evaluation of risk. Schwartz states that

> "(...) there is only one way to deal with the problem, which is firmly based on arbitrage or equilibrium in financial markets. If what the decision-maker is trying to get is the market value of the project, then, obviously, a subjective discount rate will not do the job" [Schwartz, 1994, page 1927].

[1990] and Schwartz and Smith [2000]. Another difficulty with estimating a convenience yield from a timber investment is that a stand of trees produces several different products such as lumber and paper whereas futures are traded in lumber only.

Schwartz notes that when the risk of an investment is not spanned by existing assets the value of the option should be estimated by adjusting the drift of the stochastic process for the state variable using an equilibrium model of asset prices.

An intuitive explanation for why the results of CC and DP with a constant discount rate will differ is provided in Ingersoll [1987, pages 311-313]. Trigeorgis [1996, chap 2] shows that using a constant risk adjusted discount rate implies that the market risk born per period is constant or, in other words, the total risk increases at a constant rate through time. Trigeorgis [1996] draws on the work of an earlier paper by Fama [1977] which deals with the valuation of multi-period cash flows using a Capital Asset Pricing Model (CAPM) framework. Fama [1977] shows that the correct risk adjusted discount rates implied by the CAPM model will not in general be constant, but must evolve deterministically through time. However Fama notes that the use of a constant risk adjusted discount rate may be a reasonable approximation in certain cases for "an investment project of a given type or for a firm whose activities are not anticipated to change much in nature through time" [Fama, 1977, page 23]. As is pointed out by Trigeorgis [1996], it is questionable whether this will be the case when a decision maker is faced with choices such as the potential to delay, expand, or contract an investment - i.e. in the presence of embedded options.

Although CC is judged preferable because of its better treatment of risk, it may be asked whether the DP approach is good enough in practical applications, particularly when it is difficult to obtain a reliable estimate of the convenience yield or market price of risk. In this paper we derive the condition that must hold for CC and DP, with a constant risk adjusted discount rate, to give the same result. We show that this condition will hold for certain simple cases, one of which is of particular interest because of its appearance in the literature in stylized real options models. This special case is an infinitely-lived American option with zero exercise cost and underlying state variable(s) that follows geometric Brownian motion. We argue that in more realistic real options problems, it is unwise to assume that the DP approach will give an adequate result. To demonstrate this point, we provide an example where the use of DP or CC makes a significant difference to the estimated value of a real option. The example is an optimal tree harvesting problem which has been examined previously in the literature. In this problem, the value and optimal harvest time of a stand of trees depend on the price of timber, which is assumed to be stochastic and mean reverting, and on the age of the stand.

In summary, the contributions of this paper to the literature are as follows.

- We present a proof of the conditions which must hold in order for the CC and DP approaches to give identical results. Although this proof is developed in the context of an optimal harvesting problem, it applies to a large class of real options problems in which the underlying stochastic variable follows a fairly general Ito process.

- We show that the condition for the equivalence of CC and DP will hold for some simple cases.

- We provide an example of the empirical significance of using DP versus CC in an optimal tree harvesting problem.[5] We show numerically how the risk adjusted discount rate, implied by the CC approach, changes with the stochastic state variable.

In the next section we derive the condition which must hold for CC and DP to be consistent and examine several cases for which the condition is met. In Section 3 we present the empirical example of the optimal tree harvesting problem to demonstrate that the difference between CC and DP may be significant. In Section 4 we discuss the results and lastly in Section 5 we provide some concluding comments.

2 CC and DP approaches to a real options problem

For concreteness, we use an optimal tree harvesting problem to compare the CC and DP approaches. However the resulting partial differential equations that describe the value of the option can be easily adapted to the valuation of other investment problems that depend

[5]The particular harvesting problem presented was analyzed in Insley and Lei [2007].

on a single stochastic state variable. The extension to additional stochastic state variables is also straightforward.

2.1 Dynamic Programming

In this section we describe the optimal tree harvesting model using the dynamic programming approach. The objective is to value the right to harvest a stand of trees on land that will be harvested over an infinite number of future rotations. We are using the model presented in Insley and Rollins [2005] and reproduce the details here for the convenience of the reader.

We denote the value of this asset as W, which depends on the price of timber (P), the age of the stand (α), and time (t). The price of timber is assumed to follow a known Ito process:

$$dP = a(P,t)dt + b(P,t)dz. \tag{1}$$

In Equation (1), $a(P,t)$ and $b(P,t)$ represent known functions and dz is the increment of a Wiener process.

The age of the stand, or time since the last harvest, α, is given as

$$\alpha = t - t_h \tag{2}$$

where t is the current time and t_h is the time of the last harvest. Wood volume is assumed to be a deterministic function of age:

$$Q = g(\alpha). \tag{3}$$

Age is used as a state variable, along with price, P. It follows that:

$$d\alpha = dt. \tag{4}$$

Using the dynamic programming approach the decision to harvest the stand of trees is formulated as an optimal stopping problem where the owner must decide in each period whether it is better to harvest immediately or delay until the next period. This decision process can be expressed as follows:

$$
\begin{aligned}
W(P,t,\alpha) = & max\big\{(P-C)Q + W(P,t,0); \\
& A(Q)\Delta t + (1+\rho\Delta t)^{-1}E[W(P+\Delta P, t+\Delta t, \alpha+\Delta\alpha)]\big\}
\end{aligned}
\tag{5}
$$

where

E = expectation operator

W = value of the opportunity to harvest using DP

P = price of timber

C = per unit harvesting cost

Q = current volume of timber

α = age of stand

$A(Q)$ = per period amenity value of standing forest less any management costs

ρ = risk adjusted annual discount rate

t = time.

The first expression in the curly brackets represents the return if harvesting occurs in the current period, t. It includes the net revenue from harvesting the trees plus the value of the land after harvesting, $W(P,t,0)$. This is the value that could be attained if the land were sold subsequent to the harvest, assuming that the land will remain in forestry.

The second expression in the curly brackets is the value of continuing to hold the asset (the continuation region) by delaying the decision to harvest for another period. It includes any amenity value of the standing forest, such as its value as a recreation area, less any forest management costs, $A(Q)$. In this paper amenity benefits are for simplicity set to zero so that $A(Q)$ reflects only management costs. The value in the continuation region also includes the expected value of the option to harvest in the next period, discounted to the current period. The discount rate is set exogenously to reflect the return required by an investor to hold the asset over Δt.

Following standard arguments (Dixit and Pindyck [1994], Wilmott et al. [1993]), we can derive the following partial differential equation that describes $W(P,t,\alpha)$ in the continuation region when it is optimal to delay harvesting. (Note that in Section 3.1we consider the complete problem including the value of harvesting.) We denote $W(P,t,\alpha)$ as W when there is no confusion. Subscripts t, P and α indicate partial derivatives with respect to those variables.

$$W_t + \frac{1}{2}b^2(P,t)W_{PP} + a(P,t)W_P - \rho W + A(Q) + W_\alpha = 0. \tag{6}$$

2.2 Contingent Claims

We start with the assumption that markets are sufficiently complete that project risk can be eliminated through hedging with another risky asset. We also assume that there are no arbitrage opportunities in the economy. Denoting our project of interest as V_1, we can find a traded asset, V_2, that also depends on the stochastic underlying variable P. V_2 is not the physical commodity, lumber, but rather it is a traded contract that depends on the price of lumber - perhaps the shares of a firm with harvesting rights to nearby stands of trees. By Ito's lemma V_1 and V_2 will be governed by the following stochastic processes:

$$\frac{dV_j}{V_j} = \mu_j dt + s_j dz, \quad j = 1,2 \tag{7}$$

where μ_j and s_j are functions of P, t and α. In particular,

$$\begin{aligned} \mu_j &= \left[(V_j)_t + a(P,t)(V_j)_P + (V_j)_\alpha + \frac{1}{2}b^2(P,t)(V_j)_{PP}\right]\frac{1}{V_j} \\ s_j &= \frac{b(P,t)}{V_j}(V_j)_P \end{aligned} \tag{8}$$

where

$$(V_j)_P \equiv \frac{\partial V_j}{\partial P}; (V_j)_{PP} \equiv \frac{\partial^2 V_j}{\partial P^2}; (V_j)_t \equiv \frac{\partial V_j}{\partial t}; (V_j)_\alpha \equiv \frac{\partial V_j}{\partial \alpha}. \tag{9}$$

Note that s_j is the volatility of asset j.

We can form an instantaneously riskless portfolio of V_1 and V_2 which under our no-arbitrage assumption must earn the riskfree rate of interest. Following standard arguments (see Hull [2006] for example) the following relationship will hold:

$$\frac{\mu_1 + \frac{A(Q)}{V_1} - r}{s_1} = \frac{\mu_2 - r}{s_2} \equiv \lambda_P. \tag{10}$$

μ_j is the capital gain on the contingent claim V_j. We also introduce the notation μ^T to refer to the total return on an asset from all sources. For our tree stand, $\mu_1^T = \mu_1 + \frac{A(Q)}{V}$. λ_P, called the market price of risk of P, represents the excess total return over the risk free rate per unit of variability.[6] By the no arbitrage assumption, it must be the same for all contingent claims that depend on P and t, but may vary with P and t. In Equation (10), the expression $\lambda_P s_j$ is the risk premium for contingent claim j. Dropping the j subscript for our forest stand of interest,

$$\frac{\mu + \frac{A(Q)}{V} - r}{s} = \lambda_P. \tag{11}$$

Substituting for μ and s from Equation (8) and rearranging Equation (11) we obtain the partial differential equation that must satisfy the contingent claim if it is to be held by a willing investor in the continuation region when harvesting is not optimal. (The full problem including the payout from harvesting is given in Section 3.1.

$$V_t + \frac{1}{2}b^2(P,t)V_{PP} + \left[a(P,t) - \lambda_P b(P,t)\right] V_P - rV + A(Q) + V_\alpha = 0. \tag{12}$$

According to Equation (12), we are able to value our contingent claim using the risk free rate as the discount rate, and reducing the drift rate $a(P,t)$ of the stochastic state variable by a factor $\lambda_P b(P,t)$ that reflects the extra return required to compensate for risk. Any asset dependent on P can be valued by reducing the expected growth rate of P by $\lambda_P b(P,t)$ to $[a(P,t) - \lambda_P b(P,t)]$ and discounting the resulting net benefits by the risk free rate. This result called equivalent risk neutral valuation is due to Cox et al. [1985].

The adjustment of the drift term $a(P,t)$ by λ_P results from the fact that we have hedged the price risk in V_1 with another contract, V_2. If, instead, we were able to trade lumber in

[6] Note that λ_P is an instantaneous Sharpe ratio that an asset must earn if it is subject to a specific type of risk, P risk in this case. See Cochrane [2005] for discussion of the market price of risk, the Sharpe ratio and the stochastic discount factor.

financial markets then we could hedge price risk directly by buying and selling lumber, and our hedging asset, which we will call V_3, would be $V_3 = P$. The return from holding lumber, μ_3^T, would be the sum of the capital gain, $\frac{a(P,t)}{P}$, and any convenience yield that results from holding an inventory of lumber, δ. In this case, instead of Equation (12), the fundamental partial differential equation is of the alternate form:

$$V_t + \frac{1}{2}b^2(P,t)V_{PP} + (r-\delta)PV_P - rV + A(Q) + V_\alpha = 0. \tag{13}$$

If the convenience yield is zero then we are left with the riskless rate associated with the term V_P. This is the well known result stating that if the underlying stochastic variable can be traded in financial markets represents and the convenience yield is zero then $a(P,t)$ can be replaced by the riskfree rate and there is no need to estimate a market price of risk.

In general, however, we would not expect the convenience yield to be zero, particularly for a storable commodity. Thus with the CC approach it is required that we estimate either the market price of risk or the convenience yield, which can be problematic. Both of these parameters may be non-constant. For some natural resource investments it is possible to estimate the convenience yield from futures contracts on underlying traded commodities.[7]

2.3 Comparing CC and DP

In this section we derive a necessary and sufficient condition for CC and DP to yield the same result. Let τ be defined as time remaining in the option's life, i.e. $\tau \equiv T - t$. Subtracting ρV from both sides of Equation (12), converting from t to τ, and rearranging terms, we get

$$-V_\tau + \frac{1}{2}b^2(P,\tau)V_{PP} + a(P,\tau)V_P - \rho V + V_\alpha + A(Q) = \lambda_P b(P,\tau)V_P - (\rho - r)V. \tag{14}$$

Let $Z = V - W$ where V is the known solution to Equation (14). Recall that V refers to the value of the investment using CC and W refers to the value using DP. Subtract Equation

[7]Eduardo Schwartz has done extensive work in this area. See for example Schwartz and Smith [2000], Schwartz [1998], and Schwartz [1997].

(6) (expressed in terms of τ) from Equation (14) and we obtain:

$$-Z_\tau + \frac{1}{2}b^2(P,\tau)Z_{PP} + a(P,\tau)Z_P - \rho Z + Z_\alpha = \lambda_P b(P,\tau)V_P - (\rho - r)V. \tag{15}$$

It is obvious that if $Z = 0$, the left hand side of Equation (15) will be zero implying that on the right hand side $\lambda_P b(P,\tau)V_P - (\rho - r)V = 0$. If the right hand side is zero, the risk adjusted discount rate can be expressed as

$$\rho = r + \frac{\lambda_P b(P,t)V_P}{V} = r + \lambda_P s. \tag{16}$$

Equation (16) provides a sufficient condition for DP and CC to give the same result. What is not obvious is that Equation (16) is also a necessary condition. A proof is given in Appendix A that $Z = 0$ **if and only if** Equation (16) holds. We conclude that the risk adjusted discount rate, ρ, will be constant if the volatility of the asset s (as defined in Equation (8)) and the market price of risk are constant. The asset's volatility depends on V, V_P, and $b(P,t)$ and we would not expect it to be constant except in special cases. Note that given a constant λ_P, the choice of ρ that gives a solution consistent with the CC approach requires knowledge of V, which is what we are attempting to solve for.

Using similar arguments we can extend Equation (16) to the case of two or more stochastic factors. For example, if V also depends on stochastic variable C where $dC = a'(C,t)dt + b'(C,t)dz$, the condition for DP and CC to give the same answer is:

$$\rho = r + \frac{\lambda_P b(P,t)V_P + \lambda_C b'(C,t)V_C}{V}. \tag{17}$$

where λ_C refers to the market price of risk for assets dependent on C.

As noted above, the market price of risk for a particular factor i, λ_i, may not be constant. However in many cases it is possible to estimate λ_i using the prices of contracts on traded assets which depend on the same stochastic factor. For example when valuing an investment that depends on a commodity price like oil or copper, the market price of risk or convenience yield can be estimated directly from the prices of futures contracts with varying maturities. Some researchers have used futures prices to estimate the convenience yield as an additional stochastic factor. (Schwartz [1997] is one example.) When possible it is preferable to exploit the information in futures prices to estimate the market price(s) of risk and use the CC approach, rather than use the DP approach with an assumption of an exogenous and constant risk adjusted discount rate. This view is consistent with the opinion expressed by Schwartz in the quotation given earlier (page 1). In addition, Trigeorgis [1996] argues that any approximation errors from estimating a commodity's convenience yield (or market price of risk) using futures contracts and using the estimate "to price a contingent claim on such a commodity, or a capital project involving such a commodity, will likely be far more accurate than the fruitless attempt to use a constant discount rate derived from some equilibrium model (such as CAPM) to price such a contingent claim" (page 106-107).

The situation is more equivocal when futures contracts do not exist or are not adequate (i.e. thinly traded or lacking in a variety of maturities) to provide useful information about the λ_i's for different factors. In this case estimating the market price of risk requires the use of a market equilibrium model like the capital asset pricing model and generally the λ_i's would be assumed to be constant. Undoubtedly there is error introduced in assuming a constant market price of risk just as error is introduced in assuming a constant risk adjusted discount rate with the DP approach. The appealing feature of the CC approach is that one can specify the impact of individual sources of risk through the market prices of risk in a transparent fashion. In contrast, a risk adjusted discount rate accounts for several important parameters (such as the risk free interest rate, the market prices of risk, and the volatility of the investment) in a single number. One reasonable course of action, when futures contracts are not available, is to solve for the value of the asset or project in question using a range of λ_i's. Alternatively, one may obtain upper and lower bounds on value using an uncertain market price of risk approach similar to the uncertain dividend method discussed in Wilmott [1998].[8]

[8] In a similar spirit, Cochrane [2005] demonstrates the estimation of 'good-deal bounds' for option pricing when perfect replication is impossible. The good-deal bounds "choose the market prices of risk at each instant to minimize or maximize the option price subject to a constraint that the total market price of risk

2.4 Special cases where the risk adjusted discount rate, ρ is constant

We derive special cases in which ρ will be constant so that with the appropriate choice of ρ CC and DP will give the same result. These cases are simplistic and generally not realistic for most applied real options problems. However these have been used in the literature to provide theoretical insights in stylized investment problems because an analytical solution can be obtained in these cases.

From Equation (16) we can observe that in the trivial case when the market price of risk, λ_P, is zero,then $\rho = r$ and CC and DP will give the same result. From Equation (10), we observe that $\lambda_P = 0$ implies that the total asset's total return from all sources equals the riskless rate $\mu^T = r$.[9] The market price of risk would be zero if economic agents are risk neutral or if the stochastic variables are uncorrelated with the market (i.e. no systematic risk) [Trigeorgis, 1996].

If $\lambda_P \neq 0$, then for a constant ρ we require that

$$\frac{\lambda_P b(P,t) V_P}{V} = K_1 \tag{18}$$

for some constant K_1. If we assume that λ_P is constant then we can derive a more general expression of the form that the solution to V must take to imply a constant ρ. For Equation (18) to hold, the variance rate $b(P,t)$ will need to be time invariant, so we rewrite $b(P,t)$ as $b(P)$ and write Equation (18) as:

$$\frac{dV}{V} = K_1 \frac{dP}{b(P)}. \tag{19}$$

It follows that:

$$V = e^{K_1 \int \frac{dP}{b(P)} + K_2} \tag{20}$$

with constants K_1 and K_2. When $b(P)$ takes the simple form used in this paper $b(P) = \sigma P$, V will have a solution of the form

$$V = K_3 P^{K_4} \tag{21}$$

is less than a reasonable value, compared to the Sharpe ratios of other trading opportunities."(page 347)

[9] Knudsen et al. [1999] showed the equivalence of CC and DP when $\rho = r$.

with constants K_3 and K_4.

An example of a solution of this form is V as a simple linear function of P. $K_4 = 1$ and $V = K_3P$. Substituting for V, V_P and $b(P)$ into Equation (16) gives

$$\begin{aligned}\rho &= r + \frac{\lambda_P \sigma P g}{gP} \\ &= r + \lambda_P \sigma \end{aligned} \tag{22}$$

In this case, DP with a constant discount rate as specified in Equation (22) will be consistent with the CC approach.

Equation (21) is the form of the solution for the problem presented in Dixit and Pindyck [1994, pages 136-144] which asks at what point it is optimal to pay a sunk cost I in return for a project with a value V that evolves according to geometric Brownian motion and is infinitely lived.[10] The problem presented in Dixit and Pindyck [1994] is a variation of the example in the much cited work by McDonald and Siegel [1986] which addresses a similar question, but both project value (denote as P) and investment cost (denote as C) evolve according to geometric Brownian motion:

$$\begin{aligned} dP &= \alpha_P P dt + \sigma_P P dz_P \\ dC &= \alpha_C C dt + \sigma_C C dz_C. \end{aligned} \tag{23}$$

In the McDonald and Siegel article, the firm has the opportunity to pay C_t to install an investment project with present value P_t. For an infinitely lived investment opportunity the problem is to find the boundary $B^* = \frac{P_t}{C_t}$ at which the investment should occur to maximize

$$E_0[(P_t - C_t)e^{-\rho t}] \tag{24}$$

where ρ is the discount rate and E_0 refers to the expectation at time zero. The article shows that the value of the investment can be solved for analytically and that an expression can be derived for the constant discount rate. It can be shown that the expression derived by

[10]Dixit and Pindyck denote V as the stochastic variable, and the value of the option as F.

McDonald and Siegel for the constant discount rate (their equation 10, page 716) is consistent with the more general expression for the discount rate given previously in Equation (17).

While these cases provide important intuition, more complex models are needed to address practical problems of investment under uncertainty. For example in the analysis of natural resource investments the assumption of GBM for resource prices has generally been discarded in favour of models which capture effects such mean reversion, jumps or regime switching. In addition to ongoing fixed costs, the possibility of upgrading an investment or shutting down operations are all complexities that can add realism to an economic model of investment in natural resource exploitation. Any of these factors would imply that the solution would depart from the form given in Equation (21) so that the risk adjusted discount rate would likely not be constant.

3 Empirical Example: Comparing CC and DP in an optimal harvesting problem

In this section we consider an optimal harvesting problem that was addressed in Insley and Lei [2007] using a CC approach.[11] We solve that problem using DP with a constant ρ and compare the results for asset value and optimal harvest age. We also calculate the (non-constant) values of ρ that would ensure consistency between CC and DP.

The typical tree harvesting problem will not have a simple solution such as given by Equation (21), which represents an infinitely lived asset whose value evolves according to GBM. Factors which may cause the tree harvesting problem to depart from the simple case include the presence of fixed management costs to maintain the stand, and the separate modelling of price and quantity of timber. In addition it is generally agreed that commodity prices, such as timber, are better characterized by a process that exhibits some mean reversion, as many commodity prices have been fairly flat in real terms over the long term

[11] Note that the example in Insley and Lei [2007] is similar to the case studied in Insley and Rollins [2005] with updated timber yield estimates and cost estimates.

[Schwartz, 1997]. So if we depart from the assumption of GBM prices and include other realistic characteristics, such as management costs, we would not expect that DP and CC would give the same result.

The empirical example that will be analyzed is the valuation of a stand of Jack Pine in Ontario's boreal forest. It is assumed that timber prices follow a mean reverting process of a very simple form:

$$dP = \eta(\bar{P} - P)dt + \sigma P dz. \tag{25}$$

where $\bar{P}$ is the long run average price of timber, η is the constant speed of mean reversion and σP is the variance rate.

3.1 Formulating the Variational Inequality

The tree harvesting problem is akin to an American option which can be exercised at any time. Formally it is a stochastic impulse control problem which can be solved by specifying an HJB variational inequality. We set up the HJB variational inequality for the CC version of our problem, but would proceed in a parallel fashion for the dynamic programming approach. T denotes the terminal time. Rearranging Equation (12) and substituting τ for t, we define an expression, HV, as follows:

$$HV \equiv rV - \left[\frac{1}{2}\sigma^2 P^2 V_{PP} + [a(P,\tau) - \lambda b(P,\tau)]V_P + A(Q) + V_\alpha - V_\tau\right] \tag{26}$$

In Equation (26), rV represents the return required on the investment opportunity for the risk neutral investor to continue to hold the option. The expression within square brackets represents the (certainty equivalent) return over the infinitesimal time interval $d\tau$.[12]

[12] The certainty equivalent return refers to the return on an investment when the growth rate of cash flows ($a(P,\tau)$) has been reduced by an appropriate risk premium, ($\lambda b(P,\tau)$). After this adjustment is made we can value the resulting cash flows as if investors are risk neutral.

Then the HJB variational inequality is given as:

$$\begin{aligned} &(i) \quad HV \geq 0 \\ &(ii) \quad V(P,\tau,\alpha) - [(P-C)Q + V(P,\tau,0)] \geq 0 \\ &(iii) \quad HV\Big[V(P,\tau,\alpha) - [(P-C)Q + V(P,\tau,0)]\Big] = 0 \end{aligned} \tag{27}$$

Defining $MV \equiv V(P,\tau,\alpha) - [(P-C)Q + V(P,\tau,0)]$, Equation (27) can be written compactly as

$$min[HV, MV] = 0 \tag{28}$$

Equation (27) expresses the rational individual's strategy with regards to holding versus exercising the option to harvest the stand of trees. Part (i) of Equation (27) states that the certainty equivalent return from holding the asset will be no more than the riskfree return. As long as the asset is earning the riskfree rate, it is worthwhile continuing to hold, which means delaying the harvest of the stand of trees. As the trees age and their growth rate slows, the certainty equivalent return will slip below the risk free rate, at which point it would be optimal to harvest the stand. Hence when part (i) holds as a strict equality, harvesting is delayed. When it holds as an inequality, harvesting is optimal.

Part (ii) states that the value of the option, V, must be at least as great as the return from harvesting immediately. The return from harvesting immediately is the sum of the net revenue from selling the logs, $(P-C)Q$, plus the value of the land immediately after harvesting, $V(P,t,0)$. When trees are young and growing rapidly we expect V to exceed the value of harvesting immediately (Part (ii) holds as an inequality) and it is optimal to delay harvesting the stand. As the trees age, their growth rate falls and V approaches $[(P-C)Q + V(P,\tau,0)]$. Harvesting is optimal when (ii) holds as a strict equality.

Part (iii) states that at least one of statements (i) or (ii) must hold as a strict equality. If both expressions hold as strict equalities then the investor is indifferent between harvesting and continuing to hold the asset.

The variational inequality is solved numerically which involves discretizing HV equation including a penalty term that enforces the impulse control term (Equation (27), ii). Using a fully implicit numerical scheme, we are left with a series of nonlinear algebraic equations which must be solved iteratively. Details of the solution approach are provided in Insley and Rollins [2005]

Boundary conditions can then be specified as follows.

1. **As $\boldsymbol{P} \rightarrow \mathbf{0}$**, we observe from Equation(25) no special boundary conditions are needed to prevent negative prices.

2. **As $\boldsymbol{P} \rightarrow \boldsymbol{\infty}$**, we follow Wilmott [1998] and set $V_{PP} = 0$.

3. **As $\boldsymbol{\alpha} \rightarrow \mathbf{0}$**, we require no boundary condition since the partial differential equation is first order hyperbolic in the α direction, with outgoing characteristic in the negative α direction.

4. **As $\boldsymbol{\alpha} \rightarrow \boldsymbol{\infty}$**, we assume $V_\alpha \rightarrow 0$. This means that as stand age gets very large, the value of the option to harvest, V, does not change with α. In essence we are presuming the wood volume in the stand has reached some sort of steady state.

5. **Terminal condition.** As T gets large it is assumed that $V = 0$. T is made large enough that this assumption has a negligible effect on V today.

3.2 Parameter Values: drift, diffusion, and market price of risk

We use the same values for the drift and diffusion terms of the price process as in Insley and Lei [2007]. We provide some details here (not given in Insley and Lei [2007]) on their estimation.

The historical price series used for parameter estimation is the price of spruce-pine-fir random length 2X4's in Toronto.[13] The deflated lumber price series is shown in Figure 1.[14]

[13] Data was obtained from Madison's Canadian Lumber Reporter.

[14] The original data is weekly and quoted in U.S. $ per mbf (thousand board feet). It is converted to Canadian $ per cubic metre and deflated by the Canadian consumer price index (CPI). The monthly CPI was interpolated using a cubic spline procedure to generate a weekly index.

A discrete time approximation of Equation (25) is as follows:

$$P_t - P_{t-1} = \eta \bar{P} \Delta t - \eta \Delta t P_{t-1} + \sigma P_{t-1} \sqrt{\Delta t} \epsilon_t \tag{29}$$

where ϵ_t is $N(0,1)$. We have weekly data, so $\Delta t = (1/52) year$. We performed ordinary least squares on the following equation:

$$\frac{P_t - P_{t-1}}{P_{t-1}} = c(1) + c(2) \frac{1}{P_{t-1}}. \tag{30}$$

Our estimation results are given in Table 1.[15]

The contingent claims approach requires an estimate of the market price of risk of the project, which is not directly observable. Ideally this estimate would be derived from futures markets, but lumber futures markets trade in only very short term contracts. An analysis using lumber futures is beyond the scope of this paper, and is the subject of future research. For the purposes of this paper, we will solve for stand value using a reasonable range of different values for the market price of risk.

To get a sense of what would be a reasonable value for the market price of risk, we appeal to the approach of Hull [2006, pages 716-77] which is based on the knowledge that all assets depending on the same stochastic underlying variable(s) will have the same market price of risk. An estimate can be obtained for the market price of risk for a hypothetical contract that depends linearly on the stochastic underlying variable, P. This approach is detailed in Insley and Lei [2007] and the resulting estimate for λ_P on the hypothetical contract is 0.01. For this paper, we use $\lambda_P = 0.01$ as a base case, and also consider the impact of $\lambda_P = 0.03$ and $\lambda_P = 0.05$.

[15] The estimates of η σ and $\bar{P}$ are calculated from the OLS coefficients as follows:

$$\hat{\eta} = \frac{-c(1)}{1/52}; \quad \hat{\sigma} = \frac{se}{\sqrt{1/52}}; \quad \hat{\bar{P}} = \frac{c(2)}{\hat{\eta}(1/52)} \tag{31}$$

3.3 Risk Adjusted Discount Rate

As noted above, the correct risk adjusted discount rate will, in general, change with the value of the investment. However we wish to consider the impact of a constant discount rate as is standard practice in DP applications. One way to choose a constant discount rate would be to use a risk premium consistent with Capital Asset Pricing Model. This amounts to using the expected return on the hypothetical contract, used to calculate the market price of risk for the project. The value of this contract depends linearly on price. From Equation (22)

$$\rho \equiv \mu^T = r + \lambda_P \sigma = 0.03 + 0.01 \times 0.27 = 0.0327. \tag{32}$$

Note the assumption here that the volatility of this asset is constant, hence $s = \sigma$. Similarly, for $\lambda_P = 0.03$ and 0.05, the risk adjusted discount rates are 0.0381 and 0.0435 respectively.

A more interesting comparison would be to calculate the implied risk adjusted discount rate when the market price of risk is estimated in a more sophisticated manner, such as using the Kalman filter methodology with futures contracts as in Schwartz [1997]. This is left for future research.

3.4 Timber Yield, Product Prices, and Silviculture and Harvesting Costs

The empirical example used in this paper is for a stand of Jack Pine in Ontario's boreal forest. We include a so-called basic level of silvicultural investment which represents the current level of spending on many stands in Ontario's boreal forest. Silvicultural costs[16] (in $/hectare) are $200 for site preparation and $360 to purchase nursery stock in year 1, $360 for planting in year 2, $120 for tending in year 5, and finally $10 for monitoring in year 35. Amenity value is assumed to be zero, so that A in Equation (26) reflects only silvicultural costs. The timber yield curves for Jack Pine saw logs and pulp under basic management in the boreal forest are provided in Table 2.[17]

[16]This was kindly provided by Tembec Inc.

[17]Timber yield curves were estimated by M. Penner of Forest Analysis Ltd., Huntsville,Ontario, for Tembec.

Assumptions for harvesting costs and the different log prices are given in Table 3. These prices are considered representative for 2003 prices at the millgate in Ontario's boreal forest. Average delivered wood costs to the mill for 2003 are reported as $55 per cubic meter in a recent Ontario government report [Ontario Ministry of Natural Resources, May, 2005]. From this is subtracted $8 per cubic meter as an average stumpage charge in 2003 giving $47 per cubic metre.[18] It will be noted that the lower valued items (SPF3 and poplar/birch) are harvested at a loss. These items must be harvested according to Ontario government regulation. The price for poplar/birch is at roadside, so there is no transportation cost to the mill. In the empirical application SPF1 is modelled as the key stochastic variable, with the prices of other products maintaining the same relationship with SPF1 as is shown in Table 3.

4 Empirical Results

4.1 Bare Land Value

Using the parameters described in the previous section, the HJB variational inequality, Equation (27), plus boundary conditions were solved using a fully implicit finite difference approach as describe in Insley and Rollins [2005]. We estimate the value of a stand of trees at the beginning of the first rotation (bare land value) using the CC approach and compare it with the value estimated using a DP approach with our naive risk adjusted discount rate. The results are given in Figure 2 for the three values of the market price of risk. We report values for an initial price of $60 per cubic metre for SPF1 logs.[19] [20]

[18]This consists of $35 per cubic meter for harvesting and $12 per cubic meter for transportation. Average stumpage charges are available from the Canadian Council of Forest Ministers. Land value is estimated before any stumpage charges.

[19]For the mean reverting price model, land value is very insensitive to the initial price.

[20]The accuracy of these results was checked by successive refinement of the solution grid. In addition a Richardson extrapolation was used to improve accuracy. (See Wilmott [1998] for an explanation of Richardson extrapolation.) The results indicate a numerical error of less that 1% of the value of the stand or

We observe from Figure 2 that, consistent with the theoretical discussions in Section 2, CC and DP with a constant discount rate do not give the same land values. The differences are quite significant with DP 16% below the CC value for $\lambda_P = 0.01$ and 55% below for $\lambda_P = 0.05$. Also notice that for CC, land value is quite insensitive to the tripling of the market price of risk.

If we compare the PDE's which hold in the continuation region for CC and DP, it is evident why the value computed using CC is so much larger than for DP value in the mean reverting model and also why the CC value is fairly insensitive to λ_P. To see this result, we rewrite these PDE's for convenience. For DP the relevant PDE is:

$$W_t + \frac{1}{2}b^2(P,t)W_{PP} + a(P,t)W_P - \rho W + A + W_\alpha = 0 \tag{33}$$

while for CC:

$$V_t + \frac{1}{2}b^2(P,t)V_{PP} + \left[a(P,t) - \lambda_P b(P,t)\right] V_P - rV + A + V_\alpha = 0. \tag{34}$$

Going from Equation (33) to Equation (34) the required return associated with V is reduced from ρ to the risk free rate r, which will raise the value of V, *certeris paribus.* However this is offset by the reduction of the drift rate of the stochastic price by a risk premium, which will tend to lower V. For the mean reverting model we have $a(P,t) \equiv \eta(\bar{P} - P)$ and $b(P,t) = \sigma P$, so that the term associated with V_P is $[\eta(\bar{P} - P) - \lambda_P \sigma P]$. If the speed of mean reversion η is large relative to $\lambda_P \sigma$ then the risk premium will not have a large impact on the drift term when P deviates from $\bar{P}$. The biggest effect of going from DP to CC in this case is the reduction in the discount rate to the risk free rate. Hence going from DP to CC we observe an increase in V.

4.2 Critical Harvesting Prices

Besides the value of the bare land, it is also of interest to estimate critical harvesting prices, which are determined at the point where the value of continuing to hold the option to harvest

approximately \$5 for the value of the bare land.

the stand of trees equals the value from harvesting immediately. Smooth pasting and value matching conditions hold at the critical points, although these do not need to be imposed explicitly in the numerical solution. If the current price of timber equals or exceeds the critical price for a particular stand age, then it is optimal to harvest the stand.

Critical prices for $\lambda_P = 0.01$ and 0.05 are given in Figure 3. Interestingly the critical prices track quite closely for DP and CC, remaining within 2% of each other even for the higher value of λ_P. So in this example, the apparent optimal action is basically unchanged whether CC or DP is used.

We note that critical prices start high when the trees are young and growing rapidly. It makes sense that the owner would tend to delay harvesting when the stand of trees is at a young age as there is a good economic return from leaving the trees "on the stump". As the stand ages the critical prices drop precipitously and reach a steady state of around \$85 per cubic meter after age 100. This steady state critical price seems high in relation to the long run mean reverting price of \$60 per cubic meter, particularly when the stand is at a mature age and no longer growing in volume. Several factors help to explain this seemingly high steady state critical price. First is the significant silvicultural cost for replanting which must be incurred (by regulatory requirement) as soon as the trees are harvested. Second is the fact that even though the long run mean is \$60 there is enough volatility in the price process that if the price level is below \$85 the possibility of achieving a price of \$85 or greater in the future is not insignificant, making it worthwhile delaying the harvest. For further intuition Figure 4(a) shows 500 realizations of a discrete time approximation of the price process, Equation (25), as well as a plot of the critical prices. We see that many realizations reach above the critical price curve. Figure 4(b) shows the cumulative probability of harvesting based on 5000 simulations of the price process. Specifically this graph shows the cumulative sum of the first time any realization hits the critical price or above divided by the total number of simulations. From the figure we see that by age 65 there is a 50% probability that the stand will have been harvested. This probability rises to 90% by age 100.

4.3 Implied Risk Adjusted Discount Rates

We can use our numerical results for the CC analysis to estimate the implied risk adjusted discount rate as determined by Equation (16) given our assumption of a constant market price of risk. Figure 5 (left hand graph) shows implied discount rates versus price for $\lambda_P = 0.03$. We observe that ρ varies with both price and stand age, with different curves shown for stands of different ages. The largest variation in ρ is for a stand of 50 years where ρ ranges from 3 % to about 4.7% . The comparable figure for the case where $\lambda_P = 0.01$ has a similar shape, but with a much smaller variation in ρ - from 3% to 3.5%. For $\lambda_P = 0.5$, ρ varies from 3% to 6.9%.

Further intuition may be gained from the right hand graph in Figure 5 which shows V_P/V for stands of age 35 and 50 (vertical left hand axis) and $\lambda_P \sigma P$ (vertical right hand axis). Since $\lambda_P \sigma P$ increases with P, a constant ρ would require V_P/V to decrease with P at an offsetting rate. Beyond a price of about \$150 the two are offsetting and we see that the implied risk adjusted discount rate settles at around 4 %. This makes sense since at very high prices it is optimal to harvest immediately and the ability to delay harvesting the stand has no value.

We find it informative to observe how our estimate of V_P/V changes with P, which accounts for the differing results between the CC and DP. Another interesting case would be to observe the risk adjusted discount rate with λ modelled as a stochastic factor using an approach similar to Schwartz [1997]. This is left for future research.

5 Summary and Concluding Remarks

Use of a constant risk adjusted discount rate with a dynamic programming approach is a common practice in problems of investment under uncertainty in forestry. However we have shown that a constant risk adjusted discount rate implies the value of the risky investment in question has a constant volatility over its lifetime. We presented a theorem and proof which specifies a necessary and sufficient condition for CC and DP, with a constant risk adjusted discount rate, to yield the same answer. We argued that this condition is too restrictive for most practical problems of investment under uncertainty. We presented several special cases for which the risk adjusted discount rate will be constant. One that has appeared in the literature is the case of an infinitely-lived simple American-type option in which the investment payoff follows GBM.

Using the CC approach requires the estimation of the market price of risk or convenience yield, parameters which may not be constant. For investment problems that depend on the prices of commodities traded in futures markets, these parameters can be estimated from futures prices and are sometimes modelled in the the literature as exogenous stochastic factors. In this case the CC approach is preferred to the DP approach with a constant risk adjusted discount rate, since market information can be used to provide estimates of the market price of risk for each stochastic factor.

For some investment problems, the market price of risk cannot be easily estimated using traded assets, possibly because the relevant futures markets do not exist or futures maturities are too limited. This is the case for forestry investment problems since lumber futures contracts have maturities of only up to a year.[21] We argue that even in these circumstances it is preferable to use a CC approach. The market price of risk can be estimated through an equilibrium model like the capital asset pricing model using a simple (perhaps hypothetical) contract based on the particular stochastic factor. Assuming that the market price of risk is constant will likely introduce errors into the solution. However the CC approach allows the researcher to choose the market price of risk for each individual stochastic underlying factor, rather than choosing a constant risk adjusted discount rate which combines several important parameters, including the risk free rate, market prices of risk of all stochastic factors, and the volatility of the investment asset in question. When estimating the market price of risk is problematic, a reasonable approach is to try a range of values.

We are interested in the empirical significance of using the CC versus the DP approach

[21] A solution to this problem when only short term futures contracts are available has been proposed by Schwartz [1998].

in cases which depart from the special cased when we know the two are equivalent. For illustrative purposes, we examined an optimal harvesting problem with the stochastic underlying variable following a simple mean reverting process. Other facets of this problem which depart from the special cases include the dependence of the optimal action on timber volume as well as significant replanting and silvicultural costs which must be incurred once the stand is harvested. We adopted the estimate of the constant market price of risk from Insley and Lei [2007] and then chose a range of other values for comparison. We compared investment values for CC using this constant market price of risk and DP using a constant discount rate equal to $\rho = r + \lambda_P \sigma$.[22] We found non-trivial differences ranging from 16% to 55% of land value, depending on the assumed market price of risk. However, critical harvesting prices were quite close using the two approaches. We also calculated the risk adjusted discount rates that are implied by contingent claims analysis for the forestry example. We found that the implied discount rates vary with price and stand age. Given our assumption of a constant market price of risk, this variation ranges from 0.5% point for the lowest market price of risk ($\lambda_P = 0.01$) to 3.9% points for $\lambda_P = 0.05$.

We conclude that in this optimal timber harvesting problem, the choice of DP or CC makes a significant difference empirically. This leads to the more general conclusion that for real option-type problems it is inadvisable to assume that DP and CC will be close enough so that the choice of one or the other approach is immaterial. Rather it is worth using the CC approach and taking some care in specifying the market price of risk, rather than using DP with a constant risk adjusted discount rate.

One final observation in our empirical example was that with mean reverting prices the value of the investment calculated using CC was quite insensitive to a tripling of the market price of risk. This is encouraging for real options problems that deal with natural resource commodities which are typically characterized by prices with a mean reverting property. This observation warrants further exploration using more sophisticated models of the market price

[22]Note that σ is the volatility of P whereas for the correct ρ depends on the volatility of the investment V and is not constant.

of risk or convenience yield.

A Appendix: Theorem and Proof regarding the Equivalence of CC and DP

Let $Z = V - W$ where V is the known solution to Equation (14). Recall that V refers to the value of the investment using CC and W refers to the value using DP. Subtract Equation (6) (expressed in terms of τ) from Equation (14):

$$-Z_\tau + \frac{1}{2}b^2(P,\tau)Z_{PP} + a(P,\tau)Z_P - \rho Z + Z_\alpha = \lambda_P b(P,\tau)V_P - (\rho - r)V \tag{35}$$

Let

$$\frac{1}{2}b^2(P,\tau)Z_{PP} + a(P,\tau)Z_P - \rho Z \equiv \mathcal{L}Z. \tag{36}$$

Let

$$\lambda_P b(P,\tau)V_P - (\rho - r)V \equiv f(P,\tau,\alpha). \tag{37}$$

Equation (35) can then be expressed as

$$-Z_\tau + Z_\alpha + \mathcal{L}Z = f(P,\tau,\alpha). \tag{38}$$

We assume that the boundary conditions are the same for V and W so that:

$$\begin{aligned} V(0,\tau,\alpha) &= W(0,\tau,\alpha) \\ V(P \to \infty,\tau,\alpha) &= W(P \to \infty,\tau,\alpha) \\ V(P,0,\alpha) &= W(P,0,\alpha) \end{aligned} \tag{39}$$

Equation (38) is therefore completely specified in the domain $[0,\infty] \times [0,T]$. Following from Equation (39), boundary conditions for Z are

$$\begin{aligned} Z(0,\tau,\alpha) &= 0 \\ Z(P \to \infty,\tau,\alpha) &= 0 \\ Z(P,\tau = 0,\alpha) &= 0. \end{aligned} \tag{40}$$

We can use a Green's Function to write the unique solution to Equation (38), but we must first go through some steps to deal with the term Z_α which does not contain a second derivative with respect to α.[23]

Define an arbitrary function $M(P, \tau, \theta)$ that satisfies

$$-M_\tau + \mathcal{L}M = f(P, \tau, \theta - e\tau) \tag{41}$$

where e is a constant, P and τ are as previously defined and θ is an arbitrary variable. The solution to Equation (41) can be written in terms of the Green's function, G:

$$M(P, \tau, \theta) = \int_0^\infty \int_0^\tau G(P, \tau, P', \tau') f(P', \tau', \theta - e\tau) d\tau' dP'. \tag{42}$$

We shift the function M by an amount $e\tau$ along the θ axis. Let $x = \theta - e\tau$. It follows that $M(P, \tau, x)$ satisfies

$$-M_\tau + eM_x + \mathcal{L}M = f(P, \tau, \theta). \tag{43}$$

Again using the Green's Function, the solution to (43) may be written as:

$$M(P, \tau, x) = \int_0^\infty \int_0^\tau G(P, \tau, P', \tau') f(P', \tau', \theta - e(\tau - \tau')) d\tau' dP'. \tag{44}$$

Noting the similarity between Equation (43) and Equation (38), we assume:

$$Z(P, \tau, \alpha) = M(P, \tau, x). \tag{45}$$

It then follows from Equation (44) that Z can be expressed using the Green's Function as:

$$Z(P, \tau, \alpha) = \int_0^\infty \int_0^\tau G(P, \tau, P', \tau') f(P', \tau', \alpha - e(\tau - \tau')) d\tau' dP'. \tag{46}$$

Theorem: Equivalence of DP and CC. W in Equation (6) (DP approach) and V in Equation (12) (CC approach) will be the same value (and hence $Z = 0$) if and only if $f(P, \tau, \alpha) = 0$ in $[0, \infty] \times [0, T]$

Proof. From Equation (38) we have that if $Z = 0$ then $f(P, \tau, \alpha) = 0$. Conversely from Equation (46), if $f(P', \tau', \alpha - e(\tau - \tau')) = 0$ then $Z = 0$. $\square$

[23]See Garroni and Menaldi [1992] for an explanation of the Green's Function.

From the definition of f in Equation (37) our theorem implies that $Z = 0$ so that $V = W$ if and only if $\lambda_P b(P, \tau) V_P - (\rho - r) V = 0$. For given values of λ_P, $b(P, t)$ and V, this implies that

$$\rho = r + \frac{\lambda_P b(P, t) V_P}{V}. \tag{47}$$

Parameter	Estimates (t-statistic)	Parameter	Estimates
c(1)	-.0147 (-2.71)	$\hat{\eta}$	0.8
c(2)	3.5089 (2.82)	$\hat{\bar{P}}$	\$230*
se of regression	0.0369	$\hat{\sigma}$	0.27

TABLE 1: *Parameter Estimates for Mean Reverting Price Process, Sample: weekly observations from January 1980 to July 2005. *Note that this estimated price is in \$ per cubic metre at Toronto, which had to be translated to a price at the mill gate. Since the price of \$230 dollars was close to the actual Toronto price in 2003, we adopted our estimated 2003 mill gate price of \$60 per cubic metre for SPF1 logs as* $\bar{P}$ *at the mill gate.*

Table 2: Timber volume estimates for a Jack Pine stand in Ontario's boreal forest, m^3/ha by product.

Age	NMV	SPF1	SPF2	SPF3	other[24]
1	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0
10	0.2	0.0	0.0	0.2	0.0
15	2.4	0.0	0.0	2.3	0.1
20	12.2	0.0	0.0	11.5	0.6
25	40.0	0.0	0.0	37.8	2.2
30	91.4	0.0	26.7	59.4	5.4
35	146.8	0.0	53.0	84.7	9.1
40	190.7	0.0	80.1	98.4	12.2
45	222.4	49.7	80.2	77.8	14.7
50	245.6	63.4	93.0	72.7	16.7
55	264.1	76.9	103.0	66.0	18.2
60	280.3	90.3	110.9	59.4	19.6
65	295.1	103.6	117.2	53.4	20.8
70	308.7	116.4	122.1	48.3	21.9
75	321.3	128.3	125.9	44.2	22.9
80	332.8	139.1	129.0	41.0	23.7
85	343.3	148.7	131.5	38.6	24.4
90	351.8	156.2	133.4	37.1	25.0
95	356.2	160.8	134.1	36.0	25.2
100	358.2	163.5	134.1	35.3	25.3
105	358.7	164.9	133.7	34.8	25.3

Continued on next page

Table 2 – continued from previous page

Age	NMV	SPF1	SPF2	SPF3	other
110	357.6	165.2	132.9	34.4	25.1
115	355.2	164.6	131.6	34.1	24.9
120	351.7	163.2	130.1	33.7	24.6
125	347.2	161.3	128.3	33.4	24.3
130	342.0	158.8	126.2	33.0	24.0
135	336.0	156.0	123.9	32.6	23.6
140	329.5	152.7	121.4	32.2	23.2
145	322.4	149.2	118.7	31.7	22.8
150	314.8	145.4	115.8	31.2	22.4
155	306.9	141.4	112.8	30.7	22.0
160	298.7	137.2	109.8	30.2	21.6
165	290.2	132.9	106.6	29.6	21.1
170	281.6	128.5	103.4	29.0	20.7
175	272.8	124.1	100.1	28.3	20.3
180	264.0	119.6	96.8	27.7	19.9
185	255.1	115.2	93.5	27.0	19.5
190	246.3	110.7	90.2	26.3	19.1
195	237.6	106.3	86.9	25.6	18.7
200	228.9	102.0	83.6	24.9	18.3
205	220.4	97.8	80.4	24.2	18.0
210	212.0	93.6	77.3	23.5	17.6
215	203.8	89.6	74.2	22.7	17.3
220	195.8	85.7	71.2	22.0	16.9
225	188.0	81.9	68.2	21.3	16.6

Continued on next page

Table 2 – continued from previous page

Age	NMV	SPF1	SPF2	SPF3	other
230	180.4	78.2	65.3	20.6	16.2
235	173.0	74.6	62.6	19.9	15.9
240	165.8	71.2	59.9	19.2	15.6
245	158.9	67.9	57.3	18.5	15.3
250	152.3	64.7	54.7	17.8	15.0
255	145.8	61.7	52.3	17.2	14.7

[24]NMV is net merchantable volume. SPF refers to spruce, pine, fir logs. SPF1 is greater than 16 cm diameter at the small end, SPF2 is 12 to 16 cm, and SPF3 is less than 12 cm. 'Other' refers to poplar and birch. SPF3 and 'other' are used for pulp.

Harvest and transportation cost	\$47
Price of SPF1	\$60
Price of SPF2	\$55
Price of SPF3	\$30
Price of poplar/birch	\$20

TABLE 3: *Assumed values for log prices and cost of delivering logs to the mill in \$ per cubic meter*

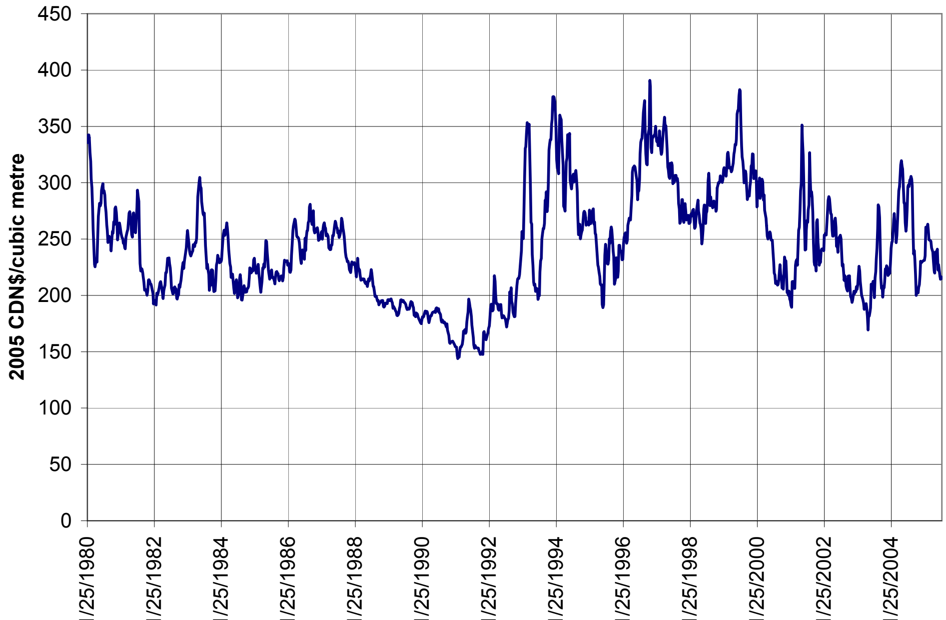


FIGURE 1: *Real price of softwood lumber, Toronto, Ontario, 2005 Canadian $ per cubic metre. (Source: Madison's Canadian Lumber Reporter, weekly data from January 1980 to July 2005, Eastern Spruce-Pine-Fir Std #2&Better, Kiln-dried, Random Length - 2x4, Deflated by the Canadian consumer price index and converted to Canadian $.)*

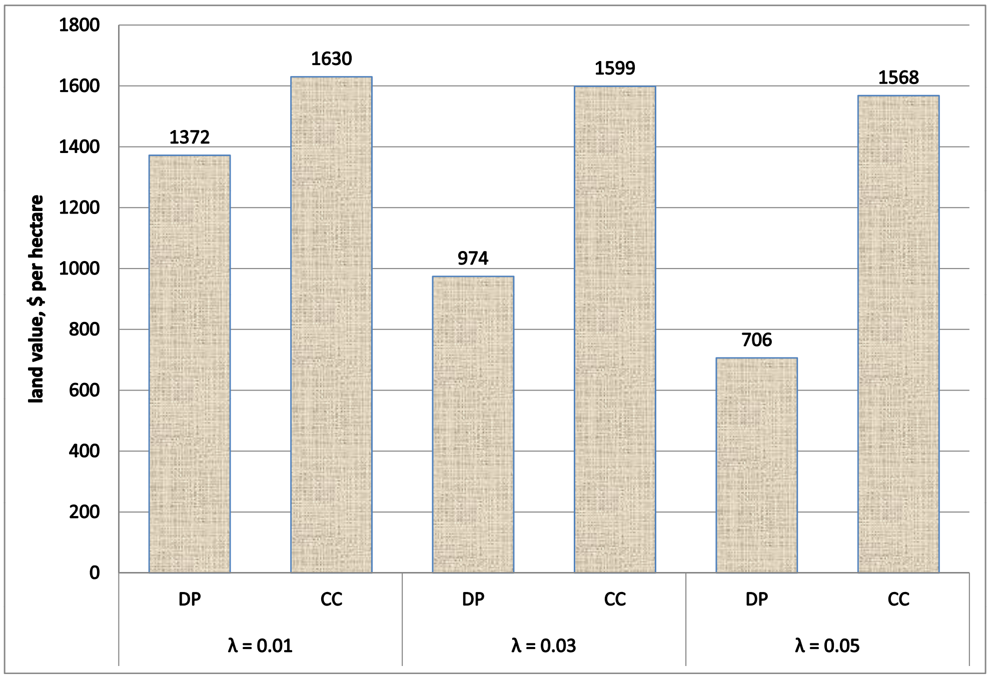
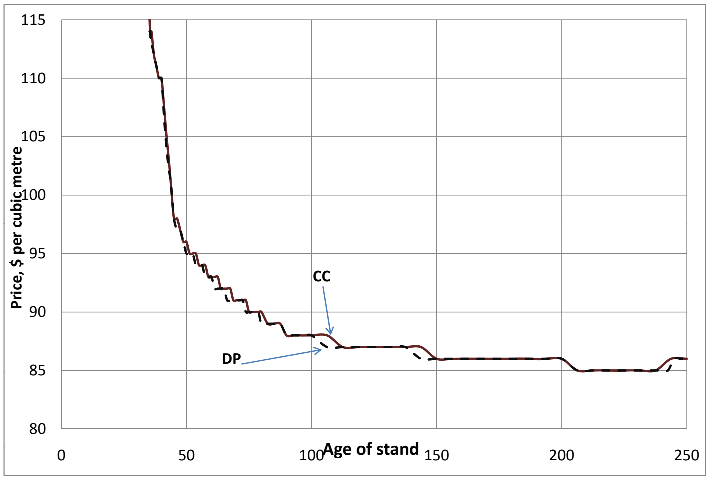
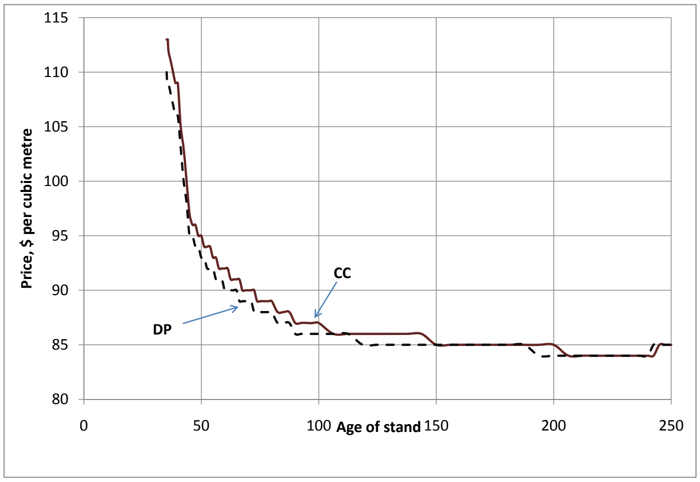


FIGURE 2: *Comparison of land values for dynamic programming (DP) and contingent claims (CC) approaches assuming an initial price of $60 per cubic metre for SPF1 logs.*

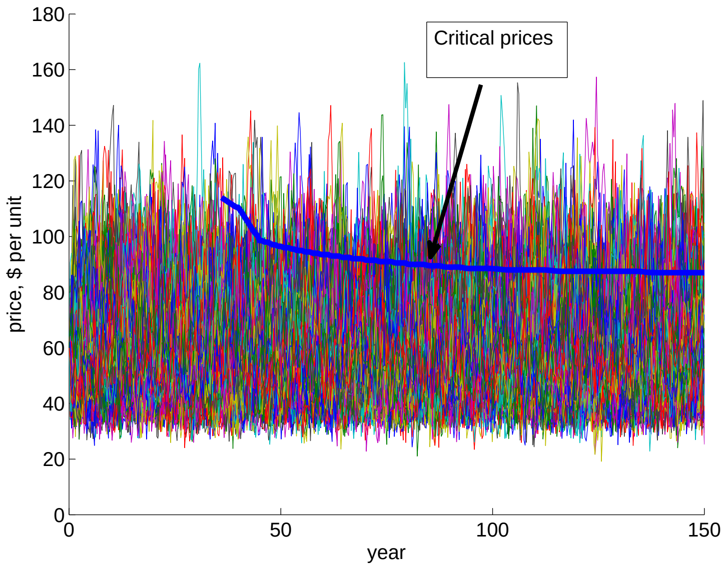


(A) $\lambda_P = 0.01$

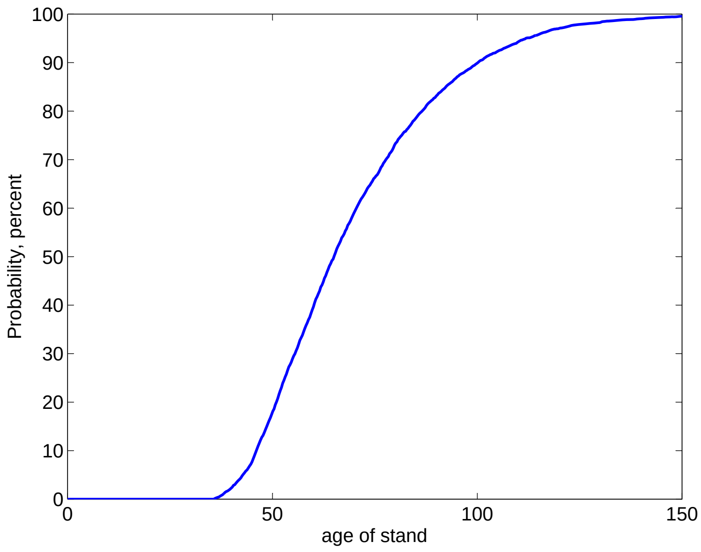


(B) $\lambda_P = 0.05$

FIGURE 3: *Critical Harvesting Prices, Comparing DP and CC for Mean Reverting Process*

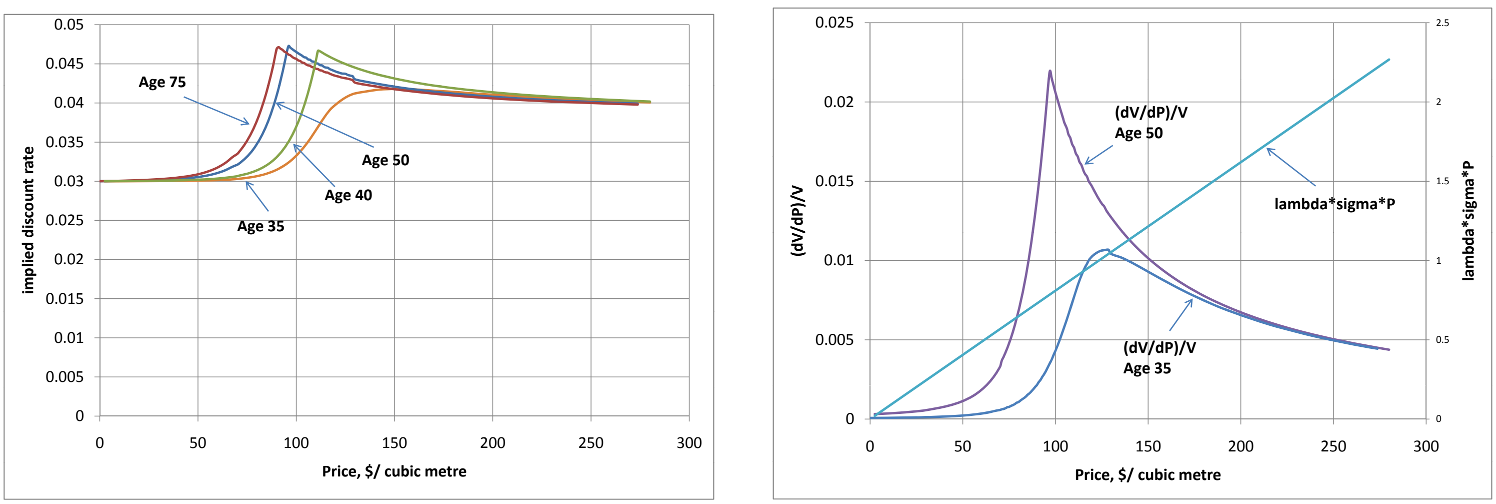


(A) *500 realizations of the price process as well as the critical harvesting prices*



(B) *Cumulative probability of harvest at each age (based on 5000 simulations)*

FIGURE 4: *Critical harvesting prices and the probability of harvesting*



(A) *Implied Discount Rate,* $\rho = r + \lambda_P \sigma P \frac{V_P}{V}$*, versus price for various stand ages.*

(B) *Components of the implied discount rate; left axis:* V_P/V *and right axis:* $\lambda_P \sigma P$*.*

FIGURE 5: *Examining the implied risk adjusted discount rate for the mean reverting process and* $\lambda_P = 0.03$

References

Luis H.R. Alvarez and Erkki Koskela. Does risk aversion accelerate optimal forest rotation under uncertainty? *Journal of Forest Economics*, 12:171–184, 2006.

Luis H.R. Alvarez and Erkki Koskela. Taxation and rotation age under stochastic forest stand value. *Journal of Environmental Economics and Management*, 54:113–127, 2007.

Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659, 1973.

M.J. Brennan and E.S. Schwartz. Evaluation of natural resource investments. *Journal of Business*, 58:135–157, 1985.

Canadian Council of Forest Ministers. Compendium of canadian forestry statistics. Technical report, Canadian Council of Forest Ministers, accessed May 2007.

H.R. Clarke and W.J. Reed. The tree-cutting problem in a stochastic environment. *Journal of Economic Dynamics and Control*, 13:569–595, 1989.

John H. Cochrane. *Asset Pricing*. Princeton University Press, 2005.

John C. Cox, Jonathan E. Ingersoll, and Stephen.A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica*, 53:363–384, 1985.

Graham Davis. Option premiums in mineral asset pricing: Are they important? *Land Economics*, 72(2):167–186, 1996.

A.K. Dixit and R.S. Pindyck. *Investment Under Uncertainty*. Princeton University Press, 1994.

E. F. Fama. Risk-adjusted discount rates and capital budgeting under uncertainty. *Journal of Financial Economics*, 35:3–34, 1977.

M.G Garroni and J.L. Menaldi. *Green Functions for Second Order Parabolic Integro-differential Equations*. Longman, 1992.

R. Gibson and E.S. Schwartz. Stochastic convenience yield and the pricing of oil contingent claims. *Journal of Finance*, 45:959–976, 1990.

Peichen Gong. Optimal harvest policy with first-order autoregressive price process. *Journal of Forest Economics*, 5(3):413–439, 1999.

R. G. Haight and T. P. Holmes. Stochastic price models and optimal tree cutting: Results for loblolly pine. *Natural Resource Modeling*, 5:423–443, 1991.

T.M. Harchaoui and Pierre Lasserre. Testing the option value theory of irreversible investment. *International Economic Review*, 42:141–166, 2001.

John C. Hull. *Options, Futures, and Other Derivatives, sixth edition*. Prentice Hall, 2006.

Jonathan E. Ingersoll. *Theory of Financial Decision Making*. Rowman and Littlefield, 1987.

Margaret Insley. A real options approach to the valuation of a forestry investment. *Journal of Environmental Economics and Management*, 44:471–492, 2002.

Margaret Insley and Manle Lei. Hedges and trees: incorporating fire risk into optimal decisions in forestry using a no-arbitrage approach. *Journal of Agricultural and Resource Economics*, 82(3):492–514, 2007.

Margaret Insley and Kimberly Rollins. On solving the multi-rotational timber harvesting problem with stochastic prices: a linear complementarity formulation. *American Journal of Agricultural Economics*, 87:735–755, 2005.

Thomas S. Knudsen, Bernhard Meister, and Mihail Zervos. On the relationship of the dynamic programming approach and the contingent claim aproach to asset valuation. *Finance and Stochastics*, 3:433–449, 1999.

J.K. Mackie-Mason. Non-linear taxation of risky assets and investment, with application to mining. *Journal of Public Economics*, 42:301–327, 1990.

Robert McDonald and Daniel Siegel. The value of waiting to invest. *Quarterly Journal of Economics*, 101(4):707–728, 1986.

Robert C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413, 1971.

Robert C. Merton. Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4(1):141–183, 1973.

R. Morck, E. Schwartz, and D. Strangeland. The valuation of forestry resources under stochastic prices and inventories. *Journal of Financial and Quantitative Analysis*, 4:473–487, 1989.

Ontario Ministry of Natural Resources. Minister's Council on Forest Sector Competitivenes, Final Report. May, 2005.

J.L Paddock, D.R. Siegel, and J.L. Smith. Option valuation of claims on real assets: The case of offshore petroleum leases. *Quarterly Journal of Economics*, 103:479–508, 1988.

A. J. Plantinga. The optimal timber rotation: An option value approach. *Forest Science*, 44:192–202, 1998.

J.-D. Saphores. The economic threshold with a stochastic pest population: A real options approach. *American Journal of Agricultural Economics*, 82(3):541–555, 2000.

E. S. Schwartz. The stochastic behaviour of commodity prices: Implications for valuation and hedging. *Journal of Finance*, 52:923–973, 1997.

Eduardo Schwartz. Review of Invesment Under Uncertainty. *The Journal of Finance*, 49(5): 1924–1928, 1994.

Eduardo Schwartz. Valuing long-term commodity assets. *Financial Management*, 27(1): 57–66, 1998.

Eduardo Schwartz and James Smith. Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46(7):893–911, 2000.

Eduardo Schwartz and Lenos Trigeorgis. *Real Options and Investment under Uncertainty*. The MIT Press, 2001.

Margaret E. Slade. Valuing managerial flexibility: An application of real-option theory to mining investments. *Journal of Environmental Economics and Management*, 41(2):193–233, 2001.

T.A. Thomson. Optimal forest rotation when stumpage prices follow a diffusion process. *Land Economics*, 68:329–342, 1992.

L. Trigeorgis. *Real Options, Managerial Flexibility and Strategy in Resource Allocations*. MIT Press, 1996.

P. Wilmott. *Derivatives, The Theory and Practice of Financial Engineering*. John Wiley & Sons, 1998.

P. Wilmott, J. Dewynne, and S. Howison. *Option Pricing, Mathematical Models and Computation*. Oxford Financial Press, 1993.

R. Yin and D. Newman. When to cut a stand of trees. *Natural Resource Modeling*, 10: 251–261, 1997.