Climate games: Who's on first? What's on second?*

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Abstract

We study four different climate change games and compare with the outcome of choices by a Social Planner. In a dynamic setting, two players choose levels of carbon emissions. Rising atmospheric carbon stocks increase average global temperature which damages player utilities. Temperature is modelled as a stochastic differential equation. We contrast the results of a Stackelberg game with a game in which both players as leaders (a Leader-Leader or Trumpian game). We also examine an Interleaved game where there is a significant time interval between player decisions. Finally we examine a game where a Nash equilibrium is chosen if it exists, and otherwise a Stackelberg game is played. One or both players may be better off in these alternative games compared to the Stackelberg game, depending on state variables. We conclude that it is important to consider alternate game structures in examining strategic interactions in pollution games. We also demonstrate that the Stackelberg game is the limit of the Interleaved game as the time between decisions goes to zero.

*The authors gratefully acknowledge funding from the Global Risk Institute, *globalriskinstitute.org*. 'Who's on first? What's on second?' is a reference to the famous comedy routine by Americans Bud Abbott and Lou Costello.

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1 Introduction

Many of the world's serious environmental problems can be described in terms of a tragedy of the commons whereby individual agents ignore the effect of their own actions on the state of particular natural assets, whether fish or forest stocks or the resilience of the world's ecosystems. The tragedy of the commons can only be alleviated by some sort of collective action, whether through government regulatory measures or through informal activities such as moral suasion at the community level. The effectiveness of actions to thwart the tragedy of the commons will depend on individual circumstances of each situation, including the strength of the incentives for individual agents to act strategically to further their own interests at the expense of the common good.

Strategic incentives related to the tragedy of the commons have long been studied in the literature using models of differential games, mostly in a deterministic setting. Long (2010) and Dockner et al. (2000) provide surveys of this large literature. Some notable contributions include Dockner & Long (1993), Zagonari (1998), Wirl (2011), List & Mason (2001). Papers tacking pollution games in a stochastic setting include Xepapadeas (1998), Nkuiya (2015), Wirl (2006). Key questions addressed are conditions for the existence of Nash equilibria, whether players are better off with cooperative behaviour, and the steady state level of pollution under cooperative versus non-cooperative games. Linear quadratic games in which utility is a quadratic function of the state variable and the state variable is linear in the control, have been used extensively as these permit a closed form solution for certain types of problems. A leading edge of the literature studies problems which include a more robust characterization of uncertainty and game characteristics such that optimal player controls may depend on state variables and are not restricted in terms of permitted strategies.

Economic models of climate change have been sharply criticized in recent years for their arbitrary assumptions regarding the costs of climate change and inadequate accounting of the uncertainly over how quickly the earth's climate will change and how human society might adapt. Pindyck (2013) is a good example of this critique. In the earlier literature, uncertainty was typically been addressed through sensitivity analysis or Monte Carlo simulation. A developing literature uses more sophisticated approaches, in particular by depicting optimal choices in fully dynamic models with explicit characterization of uncertainty in key state variables. Chesney, Lasserre & Troja (2017) examine optimal climate policies when temperature is stochastic and there is a known temperature threshold which will cause disastrous consequences if exceeded for a prolonged period of time. Other recent papers which incorporate stochasticity into one or more state variables include Crost & Traeger (2014), Ackerman, Stanton & Bueno (2013), Traeger (2014), Hambel, Kraft & Schwartz (2017).

Bressan (2011) provides an excellent summary of the specification and solution of non-cooperative differential games. He shows that in cases where the state variables evolve according to an Ito process with drift depending on player controls, value functions can be found by solving a Cauchy problem for a system of parabolic equations. The Cauchy problem is well posed if the diffusion tensor has full rank. We note that in the model studied in this paper, the diffusion tensor is not of full rank, and hence we cannot necessarily expect Nash equilibria to exist.

Insley, Snoddon & Forsyth (2018) develop a sequential pollution game model to address the specific circumstances of climate change. The model depicts two players, each being a large contributor to global carbon emissions. Players emit carbon in order to generate income, thereby increasing the atmospheric stock of carbon. Rising carbon stocks increase the average global temperature, which is modelled as an Ito process to reflect the inherent uncertainty associated with temperature. Players choose emissions in a repeated Stackelberg game. The game occurs every two years, at which time the leader and follower choose their optimal emission level, with the follower choosing immediately after the leader. There is no closed form solution to this game. A numerical approach is presented, based on the solution of a Hamilton-Jacobi-Bellman (HJB) equation.

The results of Insley, Snoddon & Forsyth (2018) indicated a classic tragedy of the commons whereby player utility is lower than would be achieved by a Social Planner seeking to maximize the sum of player utilities. Players in the game choose emission levels that are too high relative the levels chosen by a Social Planner. The paper also demonstrates the importance of temperature volatility and asymmetric damages and preferences on optimal choices. Insley, Snoddon & Forsyth (2018) do not impose the requirement that optimal strategies represent Nash equilibria. However it is possible to check for the existence of Nash equilibrium at every time step for all possible values of the state variables. This is done in the numerical example, and is reported in the paper.

The Stackelberg game has the advantage that a solution will always exist, even though the chosen optimal controls may not represent Nash equilibria. However it is reasonable to ask whether the Stackelberg game is the most appropriate for modelling climate change and other pollution games. The purpose of this paper is to examine other types of games that might be of interest in studying a pollution game. We focus, in particular on three alternatives and compare to the Stackelberg game, which we refer to as the base case. First we consider a case where both players act as leaders. In a normal Stackelberg game the leader chooses optimal emissions with the knowledge of how the follower will respond (via the follower's best response function). However it seems reasonable to ask what would happen if each player acts as a leader, mistakenly assuming the other player will respond rationally as a follower. We call this game the Leader-Leader or Trumpian scenario. To preview results, we find that in the Trumpian game, true leader (i.e. the one choosing first at time zero) is worse off than the leader in the Stackelberg game. The true follower (the player choosing second at time zero)in the Trump game is worse off than in the Stackelberg over most values of the state variables, but for certain low values of the carbon stock state variable, the follower can be better off in a Trumpian game.

In our second game variation, we focus on the time lag between the leader and follower decisions. In a case we refer to as the Interleaved game, we assume that players take turns choosing their optimal control, and there is a significant time interval between decisions. This reflects the reality that in the real world, policy decisions to change carbon emissions may take time. Again to preview our results, we find that for a medium size gap between decisions, total utility improves compared to the Stackelberg game. However, when the gap between decisions gets too large, all players are worse off.

Overall our results for the Trumpian and Interleaved games imply that if players could choose other games rather than the simple Stackelberg games, it may be in their interests to do so. We hope these results will lead to further research on decision timing and game type which will inform our understanding of strategic interactions in real world pollution games.

As noted, a focus of the pollution game literature is the characterization of Nash equilibria. To provide a comparison of the outcomes of Nash and Stackelberg controls, we examine a third game variation whereby players choose the Nash equilibrium if it exists, and otherwise revert to the optimal controls from the Stackelberg game. We refer to this case as Nash-if-Possible (or NIP). Note that about 60 percent of optimal choices in the Stackelberg game represent Nash equilibria. Our results show that the NIP and base cases are in general quite close in terms of utilities and strategies. The follower is better off in the NIP game than in the base case (pure Stackelberg game.) The leader may be better or worse off, depending on the state variables (carbon stock and temperature). Overall, however, total utility is higher under the NIP game given state variables in ranges closest to current day values.

2 Problem Formulation

This section provides an broad overview of the climate change game, which will be modelled using three different depictions of the strategic interactions of decision makers. Details of the specific games are provided in Section 3. Details of functional forms and parameter values are provided in Section 4. A summary of variable names is given in Table 1. The problem formulation is similar to that described in Insley, Snoddon & Forsyth (2018), but is repeated here for completeness of the paper.

The climate change game comprises two players each of which generate income by emitting carbon. Carbon emissions contribute to the global atmospheric stock of green house gases, which causes rising average global temperatures. Each player experiences damages from

Table 1: List of Model Variables

Variable	Description
$E_p(t)$	Emissions in region p
e_1, e_2	Particular realizations of $E_p(t)$
$S(t)$	Stock of pollution at time t, a state variable
s	A realization of $S(t)$
$\bar{S}$	preindustrial level of carbon
$\rho(t)$	Rate of natural removal of the pollution stock
$X(t)$	Average global temperature, a state variable
x	A realization of $X(t)$
$\bar{X}$	long run equilibrium level of carbon temperature
$B_p(t)$	Benefits from emissions
$C_p(t)$	Damages from pollution
π_p	Flow of net benefits to region p
r	Discount rate
$\rho(X, S, t)$	removal rate of atmospheric carbon
σ	temperature volatility
$\eta(t)$	speed of mean reversion in temperature equation

rising temperature which reduces income. Players seek to maximize their own utility through the optimal choice of per period carbon emissions, balancing the benefits from emissions with the costs that come from rising carbon stocks. And of course, the rate at which carbon stocks increase depends in part on the actions of the other player.

For simplicity we assume that there is a one to one relation between emissions and a player's income. The two players are indexed by $p = 1, 2$ and E_p refers to carbon emissions from player p. The stock of atmospheric carbon, denoted by S, is increased by emissions, but is also reduced by a natural cycle depicted by the function $\rho(X, S, t)$ and referred to as the removal rate, where X refers to average global temperature, measured in °C above preindustrial levels and t represents time. As described in Section 4, we will drop the dependence on X and S, and assume that ρ is a function only of time. Carbon stock over time is described by the stochastic differential equation:

$$\frac{dS(t)}{dt} = E_1 + E_2 + (\bar{S} - S(t))\rho(X, S, t);\ S(0) = S_0 \quad S \in [s_{min},\ s_{max}]\ . \tag{1}$$

where $\bar{S}$ is the pre-industrial equilibrium level of atmospheric carbon. Equation (1) is stochastic, in general, since the emission levels E_1, E_2, as well as possibly the decay factor ρ are in functions of stochastic state variables.

Uncertainty in the evolution of the earth's average temperature is described by an Ornstein Uhlenbeck process:

$$dX(t) = \eta(t)\left[\bar{X}(S,t) - X(t)\right]dt + \sigma dZ. \tag{2}$$

where $\eta(t)$ represents the speed of mean reversion, $\bar{X}$ represents the long run mean of global average temperature, σ is the volatility parameter, and dZ is the increment of a Wiener process.

The net benefits from carbon emissions for player p, represented by π_p are composed of the direct benefits from emissions, $B(E_p, t)$ and the damages from increasing temperature due to a growing carbon stock, $C_p(X,t)$:

$$\pi_p = B_p(E_p,t) - C_p(X,t) \quad p = 1,2; \tag{3}$$

Benefits are specified in Equation (4) as a quadratic function of emissions, which is a common assumption in the pollution game literature,

$$B_p(E_p) = aE_p(t) - E_p^2(t)/2, \quad p = 1,2;\ E_p \in [0,a], \tag{4}$$

where a is a constant. Costs of damages from climate change are specified in Equation (5) as an exponential function of temperatur,.

$$C_p(t) = \kappa_1 e^{\kappa_3 X(t)} \quad p = 1,2, \tag{5}$$

where κ_2 and κ_3 are constants.

It is assumed that the control (choice of emissions) is adjusted at fixed decision times denoted by:

$$\mathcal{T} = \{t_0 = 0 < t_1 < ...t_m... < t_M = T\}. \tag{6}$$

Let t_m^- and t_m^+ denote instants just before and after t_m, with $t_m^- = t_m - \epsilon$ and $t_m^+ = t_m + \epsilon$, $\epsilon \to 0^+$, and where T is the time horizon of interest.

$e_1^+(E_1, E_2, X, S, t_m)$ and $e_2^+(E_1, E_2, X, S, t_m)$ denote the controls implemented by the players 1 and 2 respectively, which are contained within the set of admissible controls: $e_1^+ \in Z_1$ and $e_2^+ \in Z_2$. K denotes a control set of the optimal controls for all t_m.

$$K = \left\{(e_1^+, e_2^+)_{t_0=0},\ (e_1^+, e_2^+)_{t_1=1},\ \ldots, (e_1^+, e_2^+)_{t_M=T}\right\}. \tag{7}$$

In this paper we will consider five possibilities for selection of the controls (e_1^+, e_2^+) at $t \in \mathcal{T}$: which are referred to as Stackelberg, Social Planner, Trumpian (leader-leader), Interleaved, and Nash-if-possible (NIP). We delay the precise specification of how these controls are determined until Section 3.2.

For any control strategy, the value function for player p, $V_p(e_1, e_2, x, s, t)$ is defined as:

$$V_p \quad (e_1 \quad , e_2, x, s, t) = \mathcal{E}_K \left[\int_{t'=t}^{T} e^{-rt'} \pi_p(E_1(t'), E_2(t'), X(t'), S(t')) \ \ dt' \right. \tag{8}$$
$$\left. + e^{-r(T-t)} V(E_1(T), E_2(T), \bar{X}(T), S(T), T) \ \middle| E_1(t) = e_1, E_2(t) = e_2, X(t) = x, \ S(t) = s \right] \ ,$$

where $\mathcal{E}_K[\cdot]$ is the expectation under control set K. As per convention, lower case letters e_1, e_2, x, s are used to denote realizations of the state variables E_1, E_2, X, S. The value in the final time period, T, is assumed to be the present value of a perpetual stream of expected net benefits at a given carbon stock, $S(T)$, and the long run mean temperature associated with that carbon stock level, $\bar{X}(S(T), T)$, with chosen level of emissions. This is reflected in the term $V(E_1(T), E_2(T), \bar{X}(T), S(T), T)$. The implicit assumption is that after 150 years the world has transitioned to green energy sources and emissions no longer contribute to the stock of carbon.

3 Dynamic Programming Solution

Equation (9) is solved backward in time according to the standard dynamic programming algorithm. There are two phases to the solution - for $t \in (t_m^-, t_m^+)$ we determine the optimal controls, while for $t \in (t_m^+, t_{m+1}^-)$, we solve the system of PDE's that describe how the value function changes with the evolving stock of carbon and temperature, but for fixed values of the optimal controls. As a visual aid, Equation (9) shows the noted time intervals going forward in time,

$$t_m^- \ \rightarrow \ t_m^+ \ \rightarrow \ t_{m+1}^- \ \rightarrow \ t_{m+1}^+ \ . \tag{9}$$

3.1 Advancing the solution from $t_{m+1}^- \rightarrow t_m^+$

The solution proceeds going backward in time from $t_{m+1}^- \rightarrow t_m^+$. Define the differential operator, $\mathcal{L}$ for player p, in Equation (10). The arguments in the V_p function have been suppressed when there is no ambiguity.

$$\mathcal{L}V_p \equiv \frac{(\sigma)^2}{2}\frac{\partial^2 V_p}{\partial x^2} + \eta(\bar{X}-x)\frac{\partial V_p}{\partial x} + [(e_1+e_2)+\rho(\bar{S}-s)]\frac{\partial V_p}{\partial s} - rV_p; \quad p = 1,2\ . \tag{10}$$

where r is the discount rate. Consider at time interval $h < (t_{m+1} - t_m)$. For $t \in (t_m^+, t_{m+1}^- - h)$, the dynamic programming principle states that (for small h),

$$\begin{aligned} V(e_1,e_2,s,x,t) \quad &= e^{-rh}\mathcal{E}\Big[V(E_1(t),E_2(t),S(t+h),X(t+h),t+h)\Big| \\ &S(t)=s, X(t)=x, E_1(t)=e_1, E_2(t)=e_2\Big] + \pi_p(e_1,e_2,s,x,t)h \end{aligned} \tag{11}$$

Letting $h \to 0$ and using Ito's Lemma,[1] the equation satisfied by the value function, V_p is expressed as:

$$\frac{\partial V_p}{\partial t} + \pi_p(e_1,e_2,x,s,t) + \mathcal{L}V_p = 0, \quad p = 1,2\ . \tag{12}$$

The domain of Equation (12) is $(e_1,e_2,x,s,t) \in \Omega^\infty$, where $\Omega^\infty \equiv Z_1 \times Z_2 \times [x^0,\infty] \times [\bar{S},\infty] \times [0,\infty]$. In principle, x^0 would be zero degrees Kelvin in our units. For computational purposes, we truncate the domain Ω^∞ to Ω, where $\Omega \equiv Z_1 \times Z_2 \times [x_{min}, x_{\max}] \times [s_{\min}, s_{\max}] \times [0,T]$. T, $s_{\min}$, $s_{\max}$, Z_1, Z_2, x_{min}, and $x_{\max}$ are specified based on reasonable values for the climate change problem, and are given in Section 4.

Remark 1 (Admissible sets Z_1, Z_2)**.** *We will assume in the following that Z_1, Z_2 are compact discrete sets, which would be the only realistic situation.*

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[1] Dixit & Pindyck (1994) provide an introductory treatment of optimal decisions under uncertainty characterized by an Ito process such as Equation (2). A more advanced treatment in a finance context is given by Bjork (2009).

Boundary conditions for the PDEs are specified below.

$$x \to x_{\max} \; ; \; \frac{\partial^2 V_p(e_1, e_2, x_{\max}, s, t)}{\partial x^2} = 0 \tag{13a}$$

$$x \to x_{\min} \; ; \; \sigma \to 0 \tag{13b}$$

$$s \to s_{\max} \; ; \; \frac{\partial V_P}{\partial S}(e_1 + e_2) \to 0 \tag{13c}$$

$$s \to s_{min} \; ; \; \text{No boundary condition needed, outgoing characteristics} \tag{13d}$$

$$t = T; \quad V_p = V(E_1(T), E_2(T), \bar{X}, S(T), T)/r \tag{13e}$$

The boundary at $t = T$ gives the terminal value as the the present value of an infinite stream of benefits given the long run mean temperature, $\bar{X}$, associated with the particular carbon stock and chosen emissions levels. As is described in Section 4.3, in the numerical example emissions are restricted to four possible choices. Given that emissions are no longer damaging at time T (assuming complete carbon capture and storage), the maximum possible emission level is chosen for the boundary condition. Further discussion regarding these boundary conditions can be found in Insley, Snoddon & Forsyth (2018).

More details of the numerical solution of the system of PDEs are provided in Appendix A. Suppose that the value function is decreasing in temperature at t_{m+1}^-, and that the benefits from emissions are always decreasing as a function of the temperature, then the exact value function (i.e. solution of Equation (12)) must be non-increasing in temperature at t_m^+. However, in some of our tests with extreme damage functions, this property was violated in the finite difference solution. In order to ensure this property holds for the finite difference solution, we require a mild timestep condition, as described in Appendix B.

3.2 Advancing the solution from $t_m^+ \to t_m^-$

Proceeding backwards in time, we find the optimal control in the interval between $t_m^+ \to t_m^-$. We consider several possibilities for selection of the controls (e_1^+, e_2^+) at $t \in \mathcal{T}$:

- Stackelberg;

- Social Planner;
- Leader-Leader (Trumpian);
- Interleave
- Nash-if-Possible

Recall that our controls are assumed to be feedback, i.e. a function of state. However, to avoid notational clutter in the following, we will fix $(e_1^-, e_2^-, s, x, t_m^-)$, so that, if there is no ambiguity, we will write (e_1^+, e_2^+) which will be understood to mean $(e_1^+(e_1^-, e_2^-, s, x, t_m^-), e_2^+(e_1^-, e_2^-, s, x, t_m^-))$, where e_1^- and e_2^- are the state values at t_m^- before the control is applied.

Given the optimal controls (e_1^+, e_2^+) at a point in the state space $(e_1^-, e_2^-, s, x, t_m^-)$, the dynamic programming principle implies

$$\begin{aligned} V_1(e_1^-, e_2^-, s, x, t_m^-) &= V_1(e_1^+(\cdot), e_2^+(\cdot), s, x, t_m^+) \ , \\ V_2(e_1^-, e_2^-, s, x, t_m^-) &= V_2(e_1^+(\cdot), e_2^+(\cdot), s, x, t_m^+) \ . \end{aligned} \tag{14}$$

Equation (14) is used to advance the solution backwards in time $t_m^+ \to t_m^-$, for all types of games. We describe the specific rule for determining the optimal control pair (e_1^+, e_2^+) for each type of game in the following.

3.2.1 Stackelberg Game

In the case of a Stackelberg game, suppose that, in forward time, player 1 goes first, and then player 2. Conceptually, we can then think of the time intervals (in forward time) as $(t_m^-, t_m]$, (t_m, t_m^+). Player 1 chooses control e_1^+ in $(t_m^-, t_m]$, then player 2 chooses control e_2^+ in (t_m, t_m^+).

We suppose at t_m^+, we have the value functions $V_1(e_1, e_2, s, x, t_m^+)$ and $V_2(e_1, e_2, s, x, t_m^+)$.

Definition 1 (Response set of player 2)**.** *The best response set of player 2, $R_2(\omega_1; e_2; s, x, t_m)$*

is defined to be the best response of player 2 to a control ω_1 of player 1.

$$R_2(\omega_1; e_2; s, x, t_m) \;=\; \operatorname*{argmax}_{e_2' \in Z_2} V_2(\omega_1, e_2', s, x, t_m^+) \;;\; \omega_1 \in Z_1 \;. \tag{15}$$

Remark 2 (Tie breaking)**.** *We break ties by (i) staying at the current emission level if possible, or (ii) choosing the lowest emission level. Rule (i) has priority over rule (ii). The notation $R_2(\cdot; e_2; \cdot)$ shows dependence on the state e_2 due to the tie breaking rule.*

Similarly, we define the best response set of player 1.

Definition 2 (Response set of player 1)**.** *The best response set of player 1, $R_1(\omega_2; e_1; s, x, t_m)$ is defined to be the best response of player 1 to a control ω_2 of player 2.*

$$R_1(\omega_2; e_1; s, x, t_m) \;=\; \operatorname*{argmax}_{e_1' \in Z_1} V_1(e_1', \omega_2, s, x, t_m^+) \;;\; \omega_2 \in Z_2 \;. \tag{16}$$

Ties are broken as in Remark 2. Again, to avoid notational clutter, we will fix (e_1, e_2, s, x, t_m) so that we can usually write without ambiguity $R_1(\omega_2; e_1) = R_1(\omega_2; e_1; s, x, t_m)$ and $R_2(\omega_1; e_2) = R_2(\omega_1; e_2; s, x, t_m)$.

Definition 3 (Stackelberg Game: Player 1 first)**.** *The optimal controls (e_1^+, e_2^+) assuming player 1 goes first are given by*

$$\begin{aligned} e_1^+ &= \operatorname*{argmax}_{\omega_1' \in Z_1} V_1(\omega_1', R_2(\omega_1'; e_2^-), s, x, t_m^+)\Big|_{break\ ties\ e_1^-} \;, \\ e_2^+ &= R_2(e_1^+; e_2^-) \;. \end{aligned} \tag{17}$$

3.2.2 Leader-Leader (Trumpian) Game

A leader-leader game is determined by assuming that each player (mistakenly) assumes that they are the leader. Somewhat tongue-in-cheek, we refer to this as a *Trumpian* game. The

Trumpian controls are determined from

$$
\begin{aligned}
e_1^+ &= \underset{\omega_1' \in Z_1}{\operatorname{argmax}} V_1(\omega_1', R_2(\omega_1'; e_2^-), s, x, t_m^+)\Big|_{break\ ties\ e_1^-} , \\
e_2^+ &= \underset{\omega_2' \in Z_2}{\operatorname{argmax}} V_2(R_1(\omega_2'; e_1^-), \omega_2', s, x, t_m^+)\Big|_{break\ ties\ e_2^-} .
\end{aligned} \tag{18}
$$

3.2.3 Interleave Game

Suppose that at decision times $t_{2m}; m = 0, 1, \ldots$ player one chooses an optimal control, while player two's control is fixed. At decision times $t_{2m+1}; m = 0, 1, \ldots$ player two chooses an optimal control, while player one's control is fixed. More precisely, at t_{2m}

$$
\begin{aligned}
e_1^{(2m)+} &= \text{optimal control for player 1} , \\
e_2^{(2m)+} &= e_2^{(2m)-} \ ; \ \text{player 2 control fixed} .
\end{aligned} \tag{19}
$$

At time $t_{(2m+1)}$, we have

$$
\begin{aligned}
e_1^{(2m+1)+} &= e_1^{(2m+1)-} \ ; \ \text{player 1 control fixed} , \\
e_2^{(2m+1)+} &= \text{optimal control for player 2} .
\end{aligned} \tag{20}
$$

More details for the Interleaved game are given in Appendix D. Suppose we hold player one's decision times t_{2m} fixed, and move player two's decision times t_{2m+1} to be just after t_{2m}. More precisely,

$$
t_{2m} = \text{ fixed } \ ; \ (t_{2m+1} - t_{2m}) \to 0^+ . \tag{21}
$$

In this case, intuitively, we would expect that the result of this limiting process is a Stackelberg game at times t_{2m}, with player one being the leader, and player two the follower. We confirm this intuition in Proposition 3, Appendix D.

3.2.4 Social Planner

For the Social Planner case, we have that an optimal pair (e_1^+, e_2^+) is given by

$$(e_1^+, e_2^+) = \operatorname*{argmax}_{\substack{\omega_1 \in Z_1 \\ \omega_2 \in Z_2}} \left\{ V_1(\omega_1, \omega_2, s, x, t_m^+) + V_2(\omega_1, \omega_2, s, x, t_m^+) \right\} . \tag{22}$$

Ties are broken by (i) minimizing $|V_1(e_1^+, e_2^+, s, x, t_m^+) - V_2(e_1^+, e_2^+, s, x, t_m^+)|$, (ii) choosing the lowest emission level. Rule (i) has priority over rule (ii). In other words, the Social Planner picks the emissions choices which give the most equal distribution of welfare across the two players.

3.2.5 Nash-if-Possible

In Appendix C we describe the necessary and sufficient conditions for a Nash equilibrium to exist. However, in general, we have no reason to believe that Nash equilibria exist at all points in the state space, since the system of PDEs depicted in Equation (10) is degenerate (i.e. there is no diffusion in the S direction). This observation is confirmed in our numerical tests.

In this third game for each possible combination of state variables e, e_2, x, s, we check to see whether controls e_1^+ and e_2^+ exist that represent a Nash equilibrium as defined by the necessary and sufficient conditions in Equation (17). In the event that more than one set of controls is a Nash equilibrium, then we choose the one with the lowest total emissions level. If no Nash equilibrium exists then we determine controls via a Stackelberg game as defined in Section 3.2.1.

4 Detailed model specification and parameter values

The functional forms and parameter values used in this paper are the same as in Insley, Snoddon & Forsyth (2018). For the convenience of the reader a brief review is provided in

Table 2: Base Case Parameter Values

Parameter	Description	Equation Reference	Assigned Value
$\bar{S}$	Pre-industrial atmospheric carbon stock	(1)	588 Gt carbon
s_{min}	Minimum carbon stock	(1)	588 Gt carbon
s_{max}	Maximum carbon stock	(1)	10000 Gt carbon
$\bar{\rho}$, ρ_0, ρ^*	Parameters for carbon removal Equation	(23)	0.0003, 0.01, 0.01
ϕ_1, ϕ_2, ϕ_3	Parameters of temperature Equation	(27)	0.02, 1.1817, 0.088
ϕ_4	Forcings at CO2 doubling	(25)	3.681
$F_{EX}(0)$ $F_{EX}(100)$	Parameters from forcing Equation	(25)	0.5 1
α_1, α_2	Ratio of the deep ocean to surface temp, $\alpha(t) = \alpha_1 + \alpha_2 \times t$, t is time in years with 2015 set as year 0	(27)	0.008, 0.0021
σ	Temperature volatility	(27)	0.1
x_{min}, x_{max}	Upper and lower limits on average temperature, °C	(27)	-3, 20
a_1, a_2	Parameter in benefit function, player p	(4)	10
Z_1, Z_2	Admissible controls	(7)	0, 3, 7, 10
b_1,b_2	Cost scaling parameter, players 1 & 2 respectively	(5)	15, 15
κ_1	Linear parameter in cost function for both players	(5)	0.05
κ_3	Term in exponential cost function for both players	(5)	1
T	terminal time		150 years
r	risk free rate	(10)	0.01

this section. Assumed parameter values are summarized in Table 2.

4.1 Carbon stock details

The evolution of the carbon stock is described in Equation (1). In our numerical example, we use a simplified specification of the path of carbon stock, based on Traeger (2014). We simplify the function describing the removal rate of carbon to be a deterministic function of time, denoted by $\rho(t)$, which approximates removal rates from the DICE 2016 model.

$$\rho(t) = \bar{\rho} + (\rho_0 - \bar{\rho})e^{-\rho^* t} \tag{23}$$

ρ_0 is the initial removal rate per year of atmospheric carbon, $\bar{\rho}$ is a long run equilibrium rate of removal, and ρ^* is the rate of change in the removal rate. Specific parameter assumptions for this Equation are given in Table 2. The resulting removal rate starts at 0.01 per year and falls to 0.0003 per year within 100 years.

Assumptions for the preindustrial level of carbon stock, $\bar{S}$, and the minimum and maximum carbon stock levels, s_{min} and s_{max}, are provided in Table 2. $\bar{S}$ is based on estimates used in the DICE (2016)[2] model for the year 1750. s_{max} is set at 10,000 Gt, which is well above the 6000 Gt carbon in Nordhaus (2013) and is not found to be a binding constraint in the numerical examples. A 2014 estimate of the atmospheric carbon level is 840 Gt.[3]

4.2 Stochastic process temperature: details

Equation (2) specifies the stochastic differential equation which describes temperature, $X(t)$, based on the parameters $\eta(t)$ and $\bar{X}(t)$. To relate Equation (2) to the climate change literature, we define these parameters as follows:

$$\begin{aligned} \eta(t) &\equiv \phi_1\Big(\phi_2 + \phi_3(1-\alpha(t))\Big) \\ \bar{X}(t) &\equiv \tfrac{F(S,t)}{(\phi_2+\phi_3(1-\alpha(t))}. \end{aligned} \tag{24}$$

where ϕ_1, ϕ_2, ϕ_3 and σ are constants.[4]

[2]The 2013 version of the DICE model is described in Nordhaus & Sztorc (2013). GAMS and Excel versions for the updated 2016 version are available from William Nordhaus's website: http://www.econ.yale.edu/ nordhaus/homepage/.

[3]According to the Global Carbon Project, 2014 global atmospheric CO2 concentration was 397.15 $\pm$ 0.10 ppm on average over 2014. At 2.21 Gt carbon per 1 ppm CO2, this amounts to 840 Gt carbon.(www.globalcarbonproject.org)

[4]ϕ_1, ϕ_2, ϕ_3 are denoted as ξ_1, ξ_2, and ξ_3 in Nordhaus (2013).

$F(S,t)$ refers to radiative forcing, where

$$F(S,t) = \phi_4 \left(\frac{ln(S(t)/\bar{S})}{ln(2)} \right) + F_{EX}(t) \ . \tag{25}$$

ϕ_4 indicates the forcing from doubling atmospheric carbon.[5] $F_{EX}(t)$ is forcing from causes other than carbon and is modelled as an exogenous function of time as specified in Lemoine & Traeger (2014) as follows:

$$F_{EX}(t) = F_{EX}(0) + 0.01\big(F_{EX}(100) - F_{EX}(0)\big) \min\{t, 100\} \tag{26}$$

Substituting the definitions of η and $\bar{X}$ into Equation (2) and rearranging gives

$$dX = \phi_1 \Big[F(S,t) - \phi_2 X(t) - \phi_3 [1 - \alpha(t)] X(t) \Big] dt + \sigma dZ \tag{27}$$

The drift term in Equation (27) is a simplified version of temperature models typical in Integrated Assessment Models, based on Lemoine & Traeger (2014). $\alpha(t)$ represents the ratio of the deep ocean temperature to the mean surface temperature and, for simplicity, is specified as a deterministic function of time.[6]

The values for the parameters in Equation (27) are taken from the DICE (2016) model. Note that $\phi_1 = 0.02$ which is the value reported in Dice (2016) divided by five to convert to an annual basis from the five year time steps used in the DICE (2016) model. $F_{EX}(0)$ and $F_{EX}(100)$ (Equation (25)) are also from the DICE (2016) model. The ratio of the deep ocean temperature to surface temperature, $\alpha(t)$, is modelled as a linear function of time.

4.3 Benefits and Damages

Benefits are given as a quadratic function of emissions in Equation (4). In the numerical example, there are four possible emissions levels for each player $E_p \in \{0, 3, 7, 10\}$ in gigatonnes (Gt) of carbon and we set $a_1 = a_2 = 10$ in Equation (4).

[5] ϕ_4 translates to Nordhaus's η (Nordhaus & Sztorc 2013).

[6] We are able to get a good match to the DICE2016 results using a simple linear function of time.

Damages are given as an exponential function of emissions in Equation (5). Assumed values for κ_2 and κ_3 are given in Table 2. We note that with this functional form, damages greatly exceed benefits from 3 °C onward. We view this exponential specification of damages as an alternative approach to capturing disastrous consequences, compared to adopting a Poisson jump process which is sometimes used in the literature.

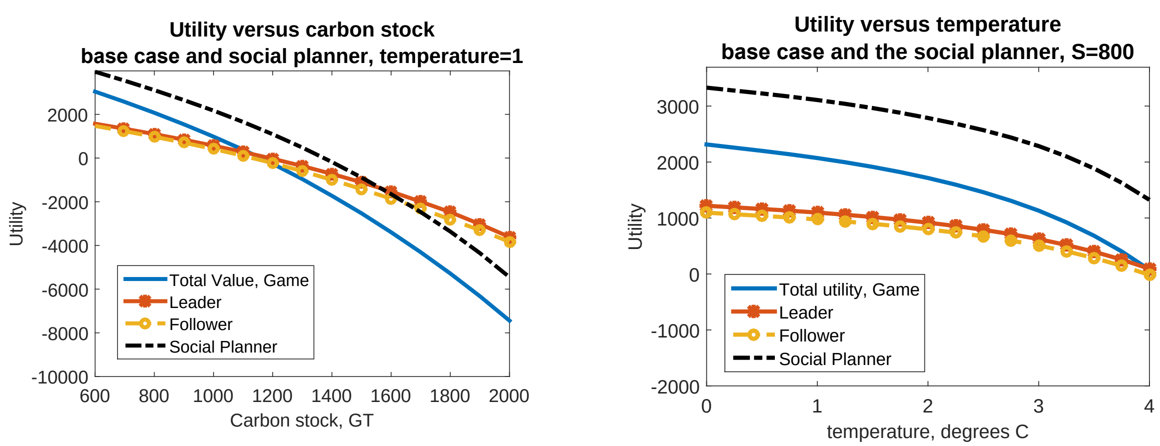
5 Numerical Results

5.1 Base case: the Stackelberg game

This section summarizes the results for the Stackelberg game which is used as the base case for comparison with other games. In this case, the leader and follower play a series of Stackelberg games at fixed decision times, set to be every two years, with the first game occurring at time zero. It is challenging to get a good sense of the results due to the numerous state variables including carbon stock, temperature, and current emission levels of each player. For the Stackelberg game, as noted in Section 3.2.1, the optimal control depends on current levels of emissions e_1 and e_2 only in the event of a tie. However, in the Interleaved case, discussed below, current emissions levels have an impact on results. We have chosen to present results for state variables close to current levels (1 °C for temperature and and 800 Gt for the atmospheric stock of carbon). We mention results for other values of state variables when this provides additional useful insight. All results are presented for time zero. For clarity when comparisons are made with other games, we will consistently refer to the leader in the Stackelberg game as Player 1 and the Follower as Player 2.

Figure 1 shows utilities for the base case game versus the Social Planner. These represent expected utility at time zero if optimal controls are followed from time zero to time T, given the dependence of the stock of carbon on the choice of emissions and given the evolution of temperature, which depends on the the carbon stock as well as a random component. Figure 1(a) plots utility versus carbon stock for a temperature of 1 °C, and for fixed state variables e_1 and e_2 both set at 10 Gt. We observe, as expected, that utility declines with carbon stock. The Social Planner case yields significantly higher utility, confirming a tragedy of the commons as an important feature of the Stackelberg game. Individual player utilities are also depicted. The leader achieves higher utility than the follower, showing that there is a benefit to being the first mover in this repeated game. At 1 °C the first mover advantage is about 10 percent, falling to zero above 5 °C. Results are depicted only for the state variable set at 1 °C, but a similar pattern emerges for other temperature levels, except that higher temperatures shift the utility curves downward.

Figure 1(b) depicts how utility changes with temperature, this time with the state variable carbon stock set at 800 Gt. (e_1 and e_2 are again set at 10 Gt, but this is immaterial in the Stackelberg case.) As expected, utility declines monotonically with increasing temperature. Again, a similar pattern emerges for plots with the stock of carbon set at different utilities, but to reduce clutter we show these graphs only for $S = 800$.



(a) Utility versus stock

(b) Utility versus temperature

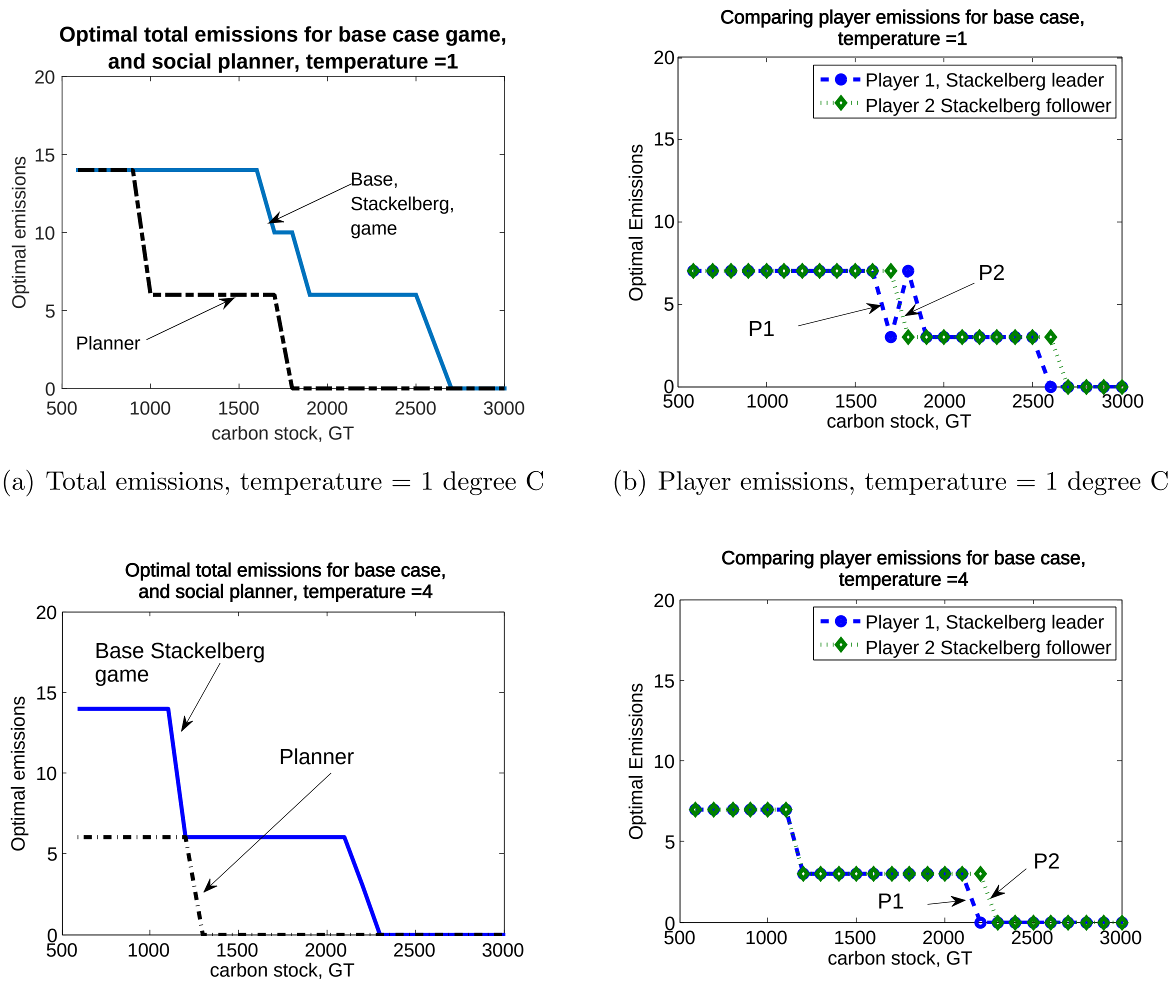
Figure 1: Utilities versus carbon stock and temperature for base Stackelberg game and Social Planner, time = 0, state variables $E1 = 10$, $E2 = 10$. Temperature is in °C above preindustrial levels.

Figure 2 compares emissions optimal choices at time zero over a range of carbon stock levels when the temperature is fixed at 1 °C (upper two graphs) and 4 °C (bottom two graphs). In Figure 2(a) and 2(c) we see that the Social Planner chooses lower emissions over most carbon stock levels compared to the total that results from the Stackelberg game. When the current temperature is at the higher level (Figure 2(c)) emissions are cut back at a lower carbon stock levels for both the game and the planner. The diagrams on the right side show that the players have largely the same strategy at time zero. In Figure 2(b) there is some see-sawing in player 1 emissions over the range $S = 1700$ to 1900. Over this range, player utilities at emission levels of 7 or 3 GT of carbon are very close together - within one percent. Given the accuracy of the numerical computation, player 1 is essentially indifferent between emissions of 3 or 7 at these points in the state space.

5.2 A Trumpian Game

We now contrast the Stackelberg game with the Leader-Leader (*Trumpian*) game, in which both players consider themselves to be the leaders in the game. Each chooses her actions assuming incorrectly that the other player will respond according to a rational best response function. (See Section 3.2.2.) In the Trump game both Player 1 and Player 2 act as leaders.

A comparison of utilities of the Trumpian and Stackelberg (base) games, and the Social Planner is given in Figure 3. The comparison shows utility versus temperature at time zero, for a fixed carbon stock $s = 800$ Gt. We observe in Figure 3(a) that the Trump game yields lower total utility than the base case Stackelberg game. The reduction is about 5% at a temperature of 1 °C, declining to zero above 5 °C. Figure 3(b) presents the results for individual players. Since players are identical and both are playing as leaders, both receive the same utilities in the Trump game. We observe Player 1 loses in this game, experiencing a significant reduction in utility (about 10 percent at 1 °C, falling to zero beyond 7 °C) compared to the Stackelberg game. Player 2 in the Trump game has a utility level that is fairly close to what is received in the Stackelberg game (1.5 percent higher in the Trump case at 1 °C). At higher temperature level, the relative benefit to Player 2 in the Trump case increases to 4 percent before declining to zero beyond 5 °C. Note that at higher levels of the carbon stock (not shown), both players are worse off in the Trump game. Under the Social Planner case both players receive higher utilities.



(a) Total emissions, temperature = 1 degree C

(b) Player emissions, temperature = 1 degree C

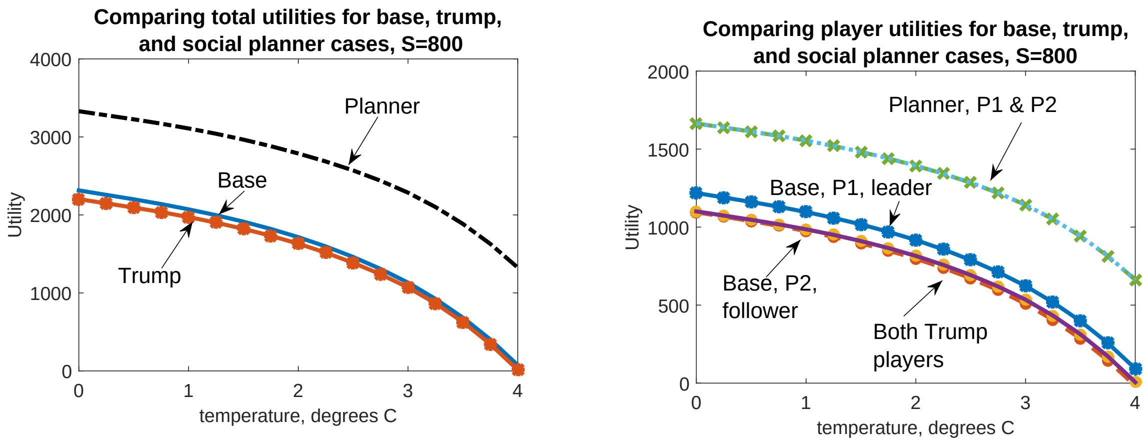
(c) Total emissions, temperature = 4 degrees C

(d) Player emissions, temperature = 4 degrees C

Figure 2: Comparing optimal controls for the base Stackelberg game and the Social Planner, time = 0. State variables $e1 = e2 =$ 10Gt. Temperature is at 1 °C and 4 °C above preindustrial levels. P1 refers to player 1, P2 refers to player 2.

It may seem counter-intuitive that over some state variables Player 2 is better off in the Trump game. This can be explained by the fact the leader is making an error in strategy at each decision point by assuming Player 2 will act as a follower. This hurts the leader and in some instances can help the follower.

Figure 4 compares the optimal controls for the Trump case with the Stackelberg game and the planner. Recall that these are optimal controls hold only $t = 0$. Future optimal



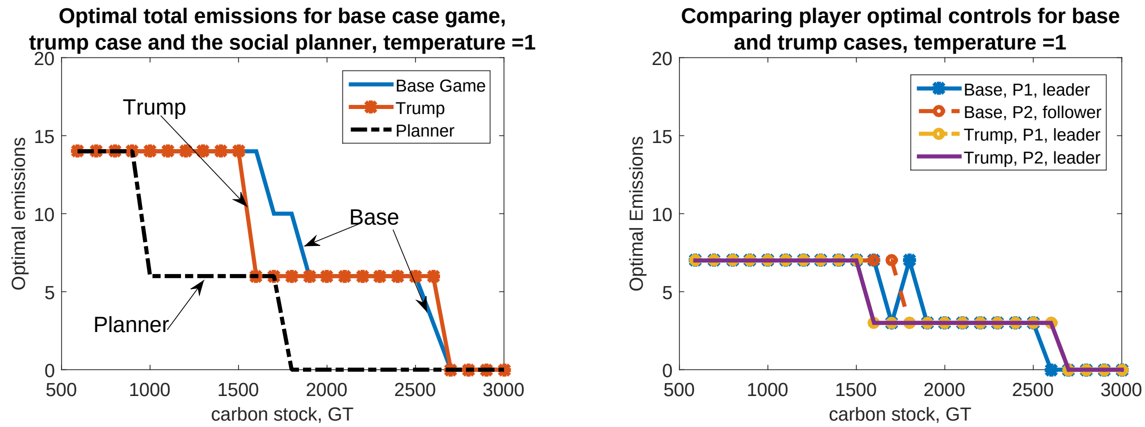
(a) Total utility

(b) Individual player utilities

Figure 3: Comparing utilities for base Stackelberg game,Trump game, and Social Planner, time $= 0$.

controls depend on the evolution of the state variables. In Figure 4(a), we observe that in the Trump game total optimal emissions are lower than the base Stackelberg game for a window of carbon stock, s, between 1600 and 1800 Gt. This is reversed over a window of high carbon stock levels (2600 - 2800 Gt) where emissions under the Trump game are higher than under the Stackelberg game. While we have not included graphs of other temperature levels, a similar pattern is observed for temperatures ranging up to 4 degrees, although the range of carbon stocks over which the Trump game has lower emissions is reduced. Figure 4(b) displays individual player optimal controls. Optimal controls for both players in the Trump game are identical. In the Stackelberg game we observe some oscillation of controls at mid carbon stock levels, which as noted early indicates the utility at these two control levels is nearly identical.

We conclude that when players are symmetric, over some levels of the state variables (lower levels for carbon stock and temperature), it is worthwhile for Player 2 (the Stackelberg follower) to be part of a Trump game. One might expect that total emissions would be higher under a Trump game over all state variables, but we can draw no such conclusion. In fact we observe that the optimal choice of emissions at time zero under the Trump game is lower



(a) Total optimal emissions

(b) Individual player optimal controls

Figure 4: Comparing optimal controls for base Stackelberg game,Trump game, and Social Planner, time = 0.

than for the Stackelberg game for certain levels of the carbon stock.

5.3 Contrasting constraints on player decision times - An Interleaved Game

In the Stackelberg game, the follower makes a choice immediately after the leader. In reality, national policies to change emissions take time to implement. This section examines a case in which there are two years between the decisions of leader and follower. This implies that each player must wait four years before choosing a new optimal control. For example, the leader makes a decision at time zero, the follower makes a decision at two years later (t=2 years), and the leader makes its next decision at two years after that (t=4 years). As is demonstrated in Section 3.2.3 and Appendix D, the Stackelberg game is the limit of the Interleaved game as the time between the leader and follower decisions goes to zero (with fixed leader decision times).

Figure 5(a) plots utility versus temperature for four different cases: the base Stackelberg game, the Trump game, the Interleaved game ($e_1 = e_2 = 10$ Gt), and the Social Planner.

Interestingly the Interleaved case shows slightly higher total utility (about 2 percent)[7] than either the Trump case or the base game. It appears that constraining each player to wait two years following the opposing player's decision before making their own choice has reduced the effect of the tragedy of the commons. Intuitively this enforced delay implies that any individual player's actions will have a more lasting effect. As an extreme, suppose player 1 is able to make decisions every two years, but player 2 is never able to take action to reduce emissions. The entire burden for reducing emissions will fall to player one. Since player two has no control available, there is by definition no tragedy of the commons.

As noted earlier, in the Interleaved game, the state variable representing current emissions affects utility. This is because there is a significant time interval before the follower (Player 2) is able to respond to the leader's (Player 1) optimal choices. At time zero, the leader goes immediately to its optimal choice, but the follower must maintain her current emissions level until two years have passed. Figure 5(b) contrasts total utility showing two different levels for player 2's current emissions, $e_2 = 0$ and $e_2 = 10$. (Player 1's current emissions are immaterial as she immediately goes to her optimal choice.) The state variable at $e_2 = 0$ gives a slightly higher total utility than when $e_2 = 10$. Note that the optimal choice of emissions for both leader and follower over this range of temperatures, and given $s = 800$ Gt, is 7 Gt.

For contrast we also include a curve labelled 'Interleave 4 year' in Figure 5(b). In this case, the time between decisions is increased to four years, so that each player can only make a choice every eight years. We see that in the four year Interleaved case, total utility is now lower than in the base game. The 'Interleave 4 year' case also has slightly lower utility than a Stackelberg game played every four years. (The 'Stackelberg 4 year' game is not shown on the graph to avoid clutter.) It is interesting that the 2 year Interleaved case (4 years between an individual player's decisions) increased utility relative to the base Stackelberg game, whereas the 4 year Interleaved case (8 years between an individual player's decisions) causes a reduction. There appears to be two countervailing effects going on. The shorter

[7] This difference depends on the stock of carbon. At $S = 1400$ and $X = 1$ °C, total utility in the interleaved game is higher by 5 percent compared to the base Stackelberg game. However for very high carbon stock levels ($S = 2200$) the difference goes to zero.

delay between decisions reduces the tragedy of the commons and increases utility, but with a longer delay this beneficial effect is overwhelmed by the negative effects of not being able to respond promptly to changes in the key state variables, temperature and carbon stock.

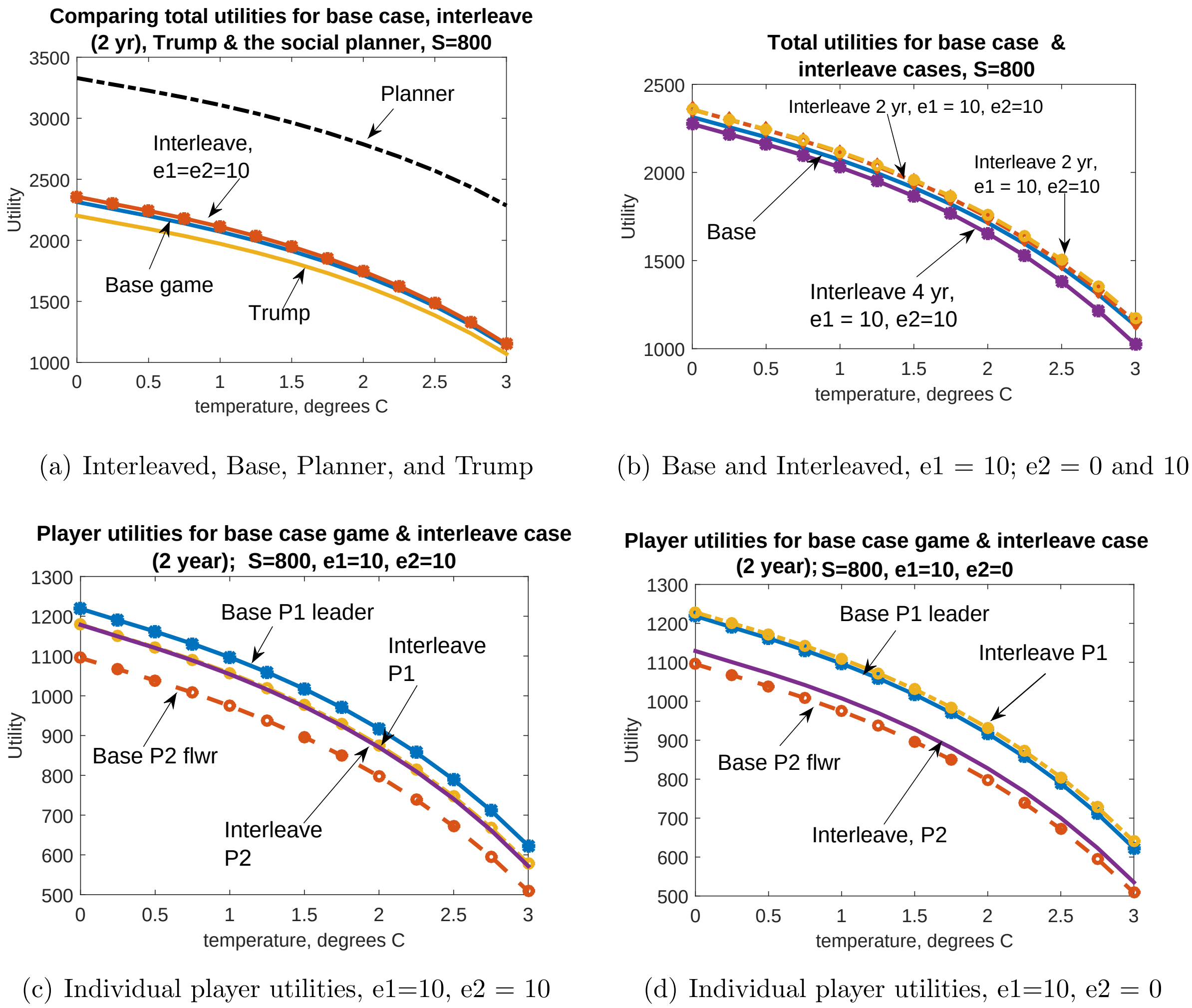
Figures 5(c) and 5(d) show the results for individual player utilities. There is some variation depending on the starting value for Player 2. The graph on the left (Figure 5(c)) shows the state variable $e_2 = 10$. Here we see Player 2 (the follower) gains from the Interleaved case relative to the base Stackelberg case, while Player 1 (the leader) is worse off. The graph on the right (Figure 5(d)) shows the state variable $e_2 = 0$. In this case, the both Player 1 and Player 2 are better off. It makes sense that the leader benefits if the follower starts the game with a very low level of emissions, which cannot be changed until 2 years later in this case.

The optimal controls for the Interleaved and base cases are shown in Figure 6. Total emissions at time zero (Figure 6(c)) are lower for the Interleaved case over a range of carbon stock levels around $S = 1800$ and $S = 2600$ Gt. Both leader and follower show different choices compared to the Stackelberg case. Compared to the Social Planner the initial choice of emissions in both games is significantly larger over a wide range of carbon stock levels.

5.4 Nash-if-possible

Our numerical computations show that Nash equilibria exist at approximately 60% of possible values for state variables, over all time steps, for the Stackelberg case. Since Nash equilibria do not always exist, we cannot do a direct comparison of Nash versus Stackelberg equilibria. However we can investigate a case were for each combination of state variables, we choose the Nash equilibrium if it exists, and if not revert to the Stackelberg game. We refer to this case as Nash-if-possible or NIP. If a Nash equilibrium does not exist, we apply the base case rules whereby player 1 goes first, and player 2 chooses immediately afterwards.

Figure 7 shows the results of this exercise. Figure 7(a) indicates that at $S = 800$ GT, total utility under NIP is slightly higher than under the base game. The difference in utility



(a) Interleaved, Base, Planner, and Trump

(b) Base and Interleaved, e1 = 10; e2 = 0 and 10

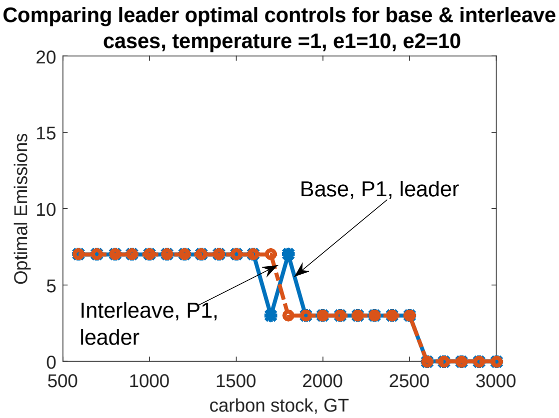
(c) Individual player utilities, e1=10, e2 = 10

(d) Individual player utilities, e1=10, e2 = 0

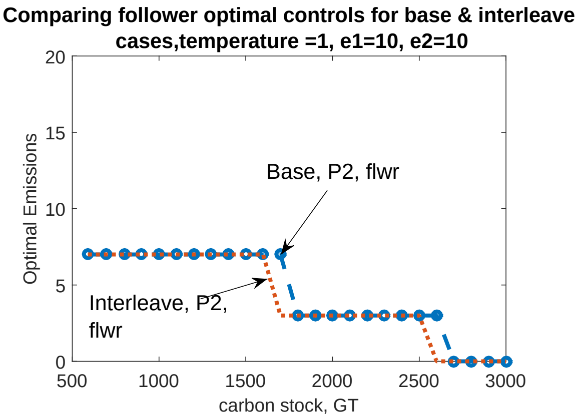
Figure 5: Comparing utilities for base Stackelberg game and Interleaved game, time = 0.

is largest at lower temperatures, and is eliminated at higher temperatures. The relative difference is 2 percent at a temperature of 0 °C, dropping to 0.5 percent at 3 °C. Figure 7(b) shows that the beneficiary of the NIP game is the follower. The leader's utility for $S = 800$ is either the same or lower than under the Stackelberg game. Figures 7(c) and 7(d) compare optimal strategies for the two games at time zero. Note that the planner chooses much lower emissions over most carbon stocks than either the base or NIP cases

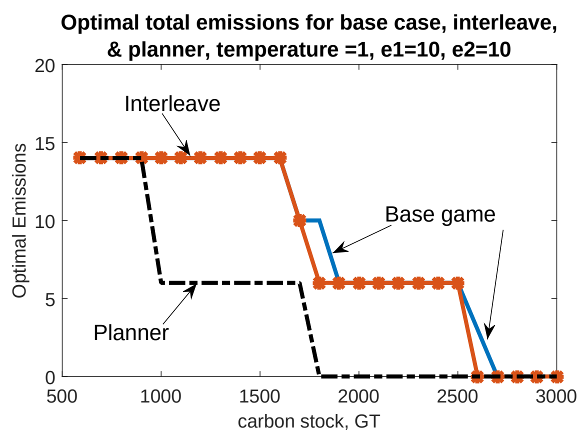
Of course the differences between the NIP and Stackelberg games change depending on current state variables. The largest differences are seen for middling carbon stock levels. For



(a) US optimal control



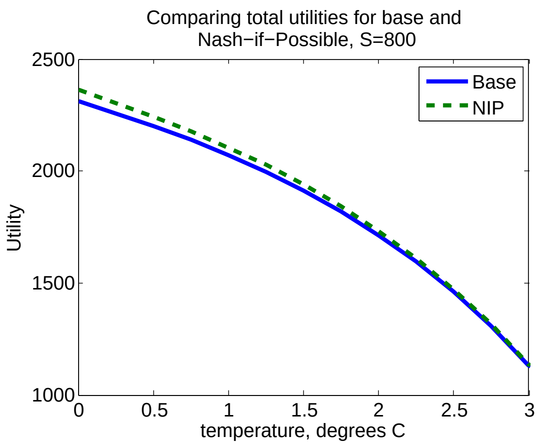
(b) CN optimal controls



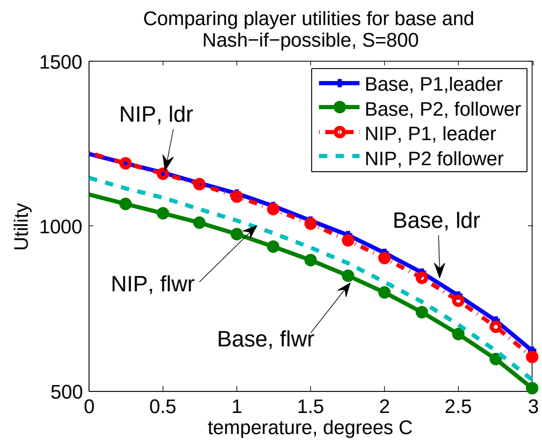
(c) Total optimal emissions

Figure 6: Comparing optimal controls for base Stackelberg game, Interleaved game, and Social Planner, time zero.

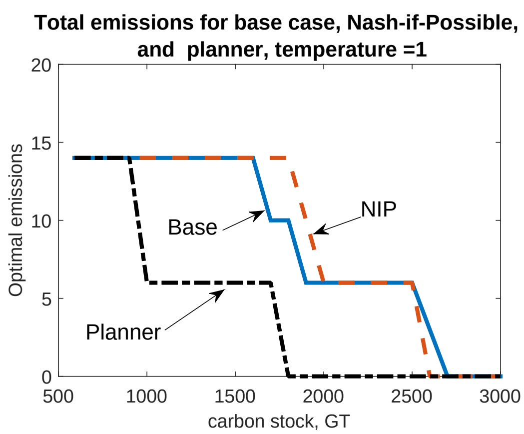
example if $S = 1400$ (not shown), total utility for NIP is higher than the base game by 5 to 12 percent at temperature levels between 1 and 3 °C. The largest beneficiary is the follower, but the leader also sees some improvement in utility.



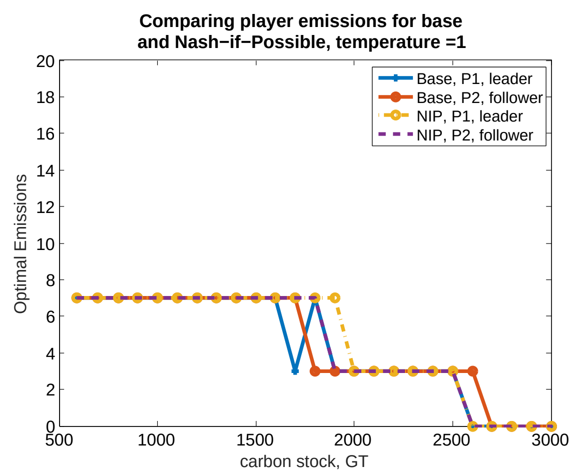
(a) Total utility: Base Case and NIP



(b) Player utilities: Base and NIP



(c) Total emissions: Base and NIP



(d) Player emissions: Base and NIP

Figure 7: Comparing utilities and emissions for Base Case and Nash-if-Possible, time = 0.

6 Concluding Comments

Strategic actions by decision makers are a key factor in our ability to confront the causes of global warming. Economic models based on game theory approaches have deepened our understanding of the consequences of strategic behaviour for the tragedy of the commons. This paper extends the pollution game literature by examining several different types of games not previously considered. We take as a starting point the differential game model of Insley, Snoddon & Forsyth (2018) which determines the closed loop optimal controls of two players choosing emission levels in a repeated Stackelberg game, while facing damages caused by rising temperatures in response to the build up of the atmospheric carbon stock. In the current paper we consider three alternative specifications of the games, which we call the Trump game, the Interleaved game, and Nash-if-Possible (NIP). These variations provide some interesting insights into the climate change game.

In the Trump game, both players act as leaders, mistakenly assuming the other player will respond rationally as a follower. Not surprisingly, total utility is lower in this game. However it is Player 1 (the leader in the Stackelberg base game) who suffers the most. At lower levels of carbon stock, Player 2 (the follower in the Stackelberg base game) actually gains slightly from the Trump game. As the carbon stock increases both players are worse off in the Trump game, but relatively speaking the leader experiences the largest reduction in utility. We conclude that in the Stackelberg game the follower might as well play like a leader, as she will be no worse off and may be better off at lower levels of the carbon stock. However the Trump game is not beneficial for the environment as total utility or welfare suffers in this game, particularly at higher carbon stock levels.

In the Interleaved game, unlike the Stackelberg game, Player 2 does not make a decision immediately after Player 1 makes her choice. Rather there is a gap of several years between player decisions. This element is intended to add some reality to the game, in that policy changes to reduce emissions do not happen instantaneously in the real world. We prove that in the limit as the time interval between player decisions goes to zero, the Interleaved game converges to the Stackelberg game.

We examined an Interleaved game of two years with a decision made by one of the players every two years, implying each player must wait four years between their own decisions. In this Interleaved game, we found that total utility increased compared to the basic Stackelberg game in which both players make optimal choices at two year intervals, with the follower choosing instantaneously after the leader. We found the follower does better in this Interleaved game compared to the Stackelberg game. The repercussions for the leader are dependent on the starting level of emissions for the follower. For low starting values for the follower, the leader also does better in the Interleaved game. However if the follower starts at high emissions levels, the leader is worse of in this Interleaved game. We interpret this result to mean that there is a benefit to a player in not reacting immediately to the actions of the other player. The follower, in particular, benefits from the fact that follower emissions cannot be changed for two years, forcing the leader to undertake any needed emissions reduction. If the follower starts with a high level of emissions, the leader is forced to react.

The relative benefits of the Interleaved game depend on the time interval between decisions. If the time between decisions is increased, eventually both players will be worse off in the Interleaved game as the extended wait between decisions does not allow the players to adequately respond to the environmental problem. We found this to be the case with an Interleaved game of four years, when individual player make decisions every eight years.

In the NIP game, we found that for lower levels of carbon stock and temperature, total utility is increased compared to the base Stackelberg game. The Stackelberg follower is the main beneficiary when both players choose a Nash equilibrium if it exists.

The Stackelberg game is convenient to apply in a differential pollution game setting, since a solution can always be found, even if optimal choices at any given time period may not be Nash. However the Stackelberg game may not be the most appropriate for the analysis of strategic decisions in certain settings. We have demonstrated three alternative games which result in improved welfare for one or both players, implying that if given the choice the players would rather be part of these alternative games. A key conclusion of our analysis is that the timing between leader and follower decisions has a crucial impact on the outcome of the game for the players, as well as for total welfare. Another interesting take-away is that the differences between the various games in terms of utility and optimal choices diminishes as temperature and/or carbon stock gets very high. The interpretation here is that when the consequences of excessive carbon emissions become dire, player strategy is no longer important as little can be done to change the outcome for any individual player.

Appendices

A Numerical methods

A.1 Advancing the solution from $t^-_{m+1} \rightarrow t^+_m$

Since we solve the PDEs backwards in time, it is convenient to define $\tau = T - t$ and use the definition

$$\begin{aligned} \hat{V}_p(e_1, e_2, x_i, s, \tau) &= V_p(e_1, e_2, x_i, s, T-\tau) \\ \hat{\pi}_p(e_1, e_2, x_i, s, \tau) &= \pi_p(e_1, e_2, x_i, s, T-\tau) \ . \end{aligned} \quad (28)$$

We rewrite Equation (12) in terms of backwards time $\tau = T - t$

$$\begin{aligned} \frac{\partial \hat{V}_p}{\partial \tau} &= \hat{\mathcal{L}}\hat{V}_p + \hat{\pi}_p + [(e_1+e_2) + \rho(\bar{S} - s)]\frac{\partial \hat{V}_p}{\partial s} \\ \hat{\mathcal{L}}\hat{V}_p &\equiv \frac{(\sigma)^2}{2}\frac{\partial^2 \hat{V}_p}{\partial x^2} + \eta(\bar{X} - x)\frac{\partial \hat{V}_p}{\partial x} - r\hat{V}_p \ . \end{aligned} \quad (29)$$

Defining the Lagrangian derivative

$$\frac{D\hat{V}_p}{D\tau} \equiv \frac{\partial \hat{V}_p}{\partial \tau} + \left(\frac{ds}{d\tau}\right)\frac{\partial \hat{V}_p}{\partial s} \ , \quad (30)$$

then Equation (29) becomes

$$\frac{D\hat{V}_p}{D\tau} = \hat{\mathcal{L}}\hat{V}_p + \pi_p \quad (31)$$

$$\frac{ds}{d\tau} = -[(e_1+e_2) + \rho(\bar{S} - s)] \ . \quad (32)$$

Integrating Equation (32) from τ to $\tau - \Delta\tau$ gives

$$s_{\tau-\Delta\tau} = s_\tau \exp(-\rho\Delta\tau) + \bar{S}(1 - \exp(-\rho\Delta\tau)) + \big(\frac{e_1+e_2}{\rho}\big)(1 - \exp(-\rho\Delta\tau)) \ . \quad (33)$$

We now use a semi-Lagrangian timestepping method to discretize Equation (29) in backwards time τ. We use a fully implicit method as described in Chen & Forsyth (2007).

$$
\begin{aligned}
\hat{V}_p(e_1, e_2, x, s_\tau, \tau) \;=\; & (\Delta\tau)\hat{\mathcal{L}}\hat{V}_p(e_1, e_2, x, s_\tau, \tau) \\
& +(\Delta\tau)\pi_p(e_1, e_2, x, s_\tau, \tau) + \hat{V}_p(e_1, e_2, x, s_{\tau-\Delta\tau}, \tau - \Delta\tau) \ . \quad (34)
\end{aligned}
$$

Equation (34) now represents a set of decoupled one-dimensional PDEs in the variable x, with (e_1, e_2, s) as parameters. We use a finite difference method with forward, backward, central differencing to discretize the $\hat{\mathcal{L}}$ operator, to ensure a positive coefficient method. (See Forsyth & Labahn (2007/2008) for details.) Linear interpolation is used to determine $\hat{V}_p(e_1, e_2, x, s_{\tau-\Delta\tau}, \tau - \Delta\tau)$. We discretize in the x direction using an unequally spaced grid with n_x nodes and in the S direction using n_s nodes. Between the time interval t^-_{m+1}, t^+_m we use n_τ equally spaced time steps. We use a coarse grid with $(n_\tau, n_x, n_s) = (2, 27, 21)$. We repeated the computations with a fine grid doubling the number of nodes in each direction to verify that the results are sufficiently accurate for our purposes.

A.2 Advancing the solution from $t^+_m \to t^-_m$

We model the possible emission levels as four discrete states for each of e_1, e_2, which gives 16 possible combinations of (e_1, e_2). We then determine the optimal controls using the methods described in Section 3.2.1. We use exhaustive search (among the finite number of possible states for (e_1, e_2)) to determine the optimal policies. This is, of course, guaranteed to obtain the optimal solution. Recall that since we use a tie-breaking rule, the optimal controls are unique.

B Monotonicity of the Numerical Solution

Economic reasoning dictates that if the value function is decreasing as a function of temperature x at $t = t^-_{m+1}$, and if the benefits are decreasing in temperature, then the value function should be decreasing in temperature at t^+_m. This can be shown to be an exact solution of

PDE (12). In our numerical tests with extreme damage functions, which resulted in rapidly changing functions π_p, we sometimes observed numerical solutions which did not have this property. In order to ensure that this desirable property of the solution holds, we require a timestep restriction. To the best of our knowledge, this restriction has not been reported previously. In practice, this restriction is quite mild, but nevertheless important for extreme cases.

We remind the reader that since we solve the PDEs backwards in time, it is convenient to use the definitions

$$
\begin{aligned}
\hat{V}_p(e_1, e_2, x_i, s, \tau) &= V_p(e_1, e_2, x_i, s, T - \tau) \\
\hat{\pi}_p(e_1, e_2, x_i, s, \tau) &= \pi_p(e_1, e_2, x_i, s, T - \tau) \ .
\end{aligned} \tag{35}
$$

Assuming that we discretize Equation (34) on a finite difference grid $x_i, i = 1, \dots, n_x$, we define

$$
\begin{aligned}
V_i^{n+1} &= \hat{V}_p(e_1, e_2, x_i, s_{\tau^{n+1}}, \tau^{n+1}) \\
c_i \equiv c(x_i) &= \hat{\pi}_p(e_1, e_2, x_i, s_{\tau^{n+1}}, \tau^{n+1})\Delta\tau + \hat{V}_p(e_1, e_2, x_i, s_{\tau^n}, \tau^n)
\end{aligned} \tag{36}
$$

Using the methods in Forsyth & Labahn (2007/2008), we discretize Equation (34) using the definitions (36) as follows

$$
-\alpha_i \Delta\tau V_{i-1}^{n+1} + (1 + (\alpha_i + \beta_i + r)\Delta\tau) V_i^{n+1} - \beta_i \Delta\tau V_{i+1}^{n+1} = c_i \ , \tag{37}
$$

for $i = 1, \dots, n_x$. Note that the boundary conditions used (see Section 3.1) imply that $\alpha_1 = 0$ and that $\beta_{n_x} = 0$, so that Equation (37) is well defined for all $i = 1, \dots, n_x$. See Forsyth & Labahn (2007/2008) for precise definitions of α_i and β_i.

Note that by construction α_i, β_i satisfy the positive coefficient condition

$$
\alpha_i \geq 0 \quad ; \quad \beta_i \geq 0 \quad ; \quad i = 1, \dots, n_x \ . \tag{38}
$$

Assume that

$$\begin{aligned} \hat{V}_p(e_1, e_2, x_{i+1}, s_{\tau^n}, \tau^n) - \hat{V}_p(e_1, e_2, x_i, s_{\tau^n}, \tau^n) &\leq 0 \\ \hat{\pi}_p(e_1, e_2, x_{i+1}, s_{\tau^{n+1}}, \tau^{n+1}) - \hat{\pi}_p(e_1, e_2, x_i, s_{\tau^{n+1}}, \tau^{n+1}) &\leq 0 \ , \end{aligned} \tag{39}$$

which then implies that

$$c_{i+1} - c_i \leq 0 \ . \tag{40}$$

If Equation (40) holds, then we should have that $V_{i+1}^{n+1} - V_i^{n+1} \leq 0$ (this is a property of the exact solution of Equation (34) if $c(y) - c(x) \leq 0$ if $y > x$).

Define $U_i = V_{i+1}^{n+1} - V_i^{n+1}, i = 1, \ldots, n_x - 1$. Writing Equation (37) at node i and node $i + 1$ and subtracting, we obtain the following Equation satisfied by U_i,

$$\begin{aligned} [1 + \Delta\tau(r + \alpha_{i+1} + \beta_i)]U_i - \Delta\tau\alpha_i U_{i-1} - \Delta\tau\beta_{i+1}U_{i+1} &= \Delta\tau(c_{i+1} - c_i) \\ & i = 1, \ldots, n_x - 1 \\ & \alpha_1 = 0 \ ; \ \beta_{n_x} = 0 \ . \end{aligned} \tag{41}$$

Let $U = [U_1, U_2, \ldots, U_{nx-1}]'$, $B_i = \Delta\tau(c_{i+1} - c_i)$, $B = [B_1, B_2, \ldots, B_{nx-1}]'$. We can then write Equation (41) in matrix form as

$$QU = B \ , \tag{42}$$

where

$$\big[QU\big]_i = [1 + \Delta\tau(r + \alpha_{i+1} + \beta_i)]U_i - \Delta\tau\alpha_i U_{i-1} - \Delta\tau\beta_{i+1}U_{i+1} \ . \tag{43}$$

Recall the definition of an M matrix (Varga 2009),

Definition 4 (Non-singular M-matrix)**.** *A square matrix Q is a non-singular M matrix if (i) Q has non-positive off-diagonal elements (ii) Q is non-singular and (iii) $Q^{-1} \geq 0$.*

A useful result is the following (Varga 2009)

Theorem 1. *A sufficient condition for a square matrix Q to be a non-singular M matrix is that (i) Q has non-positive off-diagonal elements (ii) Q is strictly row diagonally dominant.*

From Theorem 1, and Equation (43), a sufficient condition for Q to be an M matrix is that

$$1 + \Delta\tau[r + (\alpha_{i+1} - \alpha_i) + (\beta_i - \beta_{i+1})] > 0 \ , \ i = 1, \ ... \ n_{x-1} \tag{44}$$

which for a fixed temperature grid, can be satisfied for a sufficiently small $\Delta\tau$. If $\min_i(x_{i+1} - x_i) = \Delta x$, then $\alpha_i = O((\Delta x)^{-2})$, $\beta_i = O((\Delta x)^{-2})$. If α_i, β_i are smoothly varying coefficients, then we can assume that

$$|\alpha_{i+1} - \alpha_i| = O\left(\frac{1}{\Delta x}\right) \quad ; \quad |\beta_i - \beta_{i+1}| = O\left(\frac{1}{\Delta x}\right) \ , \tag{45}$$

and hence condition (44) is essentially a condition on $\Delta\tau/\Delta x$. In practice, for smoothly varying coefficients, $|\alpha_{i+1} - \alpha_i|$ and $|\beta_i - \beta_{i+1}|$ are normally small, so the timestep condition (44) is quite mild.

Proposition 1 (Monotonicity result)**.** *Suppose that (i) condition (44) is satisfied and (ii) $B_i = \Delta\tau(c_{i+1} - c_i) \leq 0$, then $U_i = V_{i+1}^{n+1} - V_i^{n+1} \leq 0$.*

Proof. From condition (44), Definition 4, and Theorem 1 we have that $Q^{-1} \geq 0$, hence from Equation (42)

$$U \ = \ Q^{-1}B \leq 0 \ . \tag{46}$$

□

The practical implication of this result is that if conditions (39) hold at $\tau = T - t_{m+1}^-$, then $\hat{V}(\cdot, \tau = T - t_m^+)$ is a non increasing function of temperature. However, this property may be destroyed after application of the optimal control at $\tau = T - t_m^+ \to T - t_m^-$. In other words, if we observe that the solution is increasing in temperature, this can only be a result of applying the optimal control, and is not a numerical artifact.

C Nash Equilibrium

We again fix (e_1, e_2, s, x, t_m), so that we understand that $e_p^+ = e_p^+(e_1, e_2, s, x, t_m)$, $R_p(\omega; e_1^-) = R_p(\omega; e_p^-; s, x, t_m)$.

Definition 5 (Nash Equilibrium)**.** *Given the best response sets $R_2(\omega_1; e_2^-)$, $R_1(\omega_2; e_1^-)$ defined in Equations (15)-(16), then the pair (e_1^+, e_2^+) is a Nash equilibrium point if and only if*

$$e_1^+ = R_1(e_2^+; e_1^-) \quad ; \quad e_2^+ = R_2(e_1^+; e_2^-) \ . \tag{47}$$

The following proposition is proven in Insley, Snoddon & Forsyth (2018).

Proposition 2 (Sufficient condition for a Nash Equilibrium)**.** *Suppose $(\hat{e}_1^+, \hat{e}_2^+)$ is the Stackelberg control if player 1 goes first and $(\bar{e}_1^+, \bar{e}_2^+)$ is the Stackelberg control if player 2 goes first. A Nash equilibrium exists at a point (e_1, e_2, s, x, t_m) if $(\hat{e}_1^+, \hat{e}_2^+) = (\bar{e}_1^+, \bar{e}_2^+)$.*

Remark 3 (Checking for a Nash equilibrium)**.** *A necessary and sufficient condition for a Nash Equilibrium is given by condition* (47)*. However a sufficient condition for a Nash equilibrium in the Stackelberg game is that optimal control of either player is independent of who goes first.*

D Interleave Game

In this appendix, we consider the situation where each player makes optimal decisions alternatively. These decision times are separated by a finite time interval.

Suppose that player one chooses an optimal control at time t_m, which we denote by e_1^{m+}. Player two's control is fixed at the value e_2^{m-}. At time t_{m+1}, player two chooses a control

$e_2^{(m+1)+}$, while player one's control is fixed at $e_1^{(m+1)-}$. To avoid notational clutter, we will fix the state variables (s, x) in the following, with the dependence on (s, x) understood.

At time t_m, we have, with player two's control fixed at e_2^{m-},

$$\begin{aligned} V_1(e_1^{m-}, e_2^{m-}, t_m^-) &= V_1(e_1^{m+}, e_2^{m-}, t_m^+) \quad (48)\\ V_2(e_1^{m-}, e_2^{m-}, t_m^-) &= V_2(e_1^{m+}, e_2^{m-}, t_m^+) \ . \quad (49)\end{aligned}$$

Player one's control is determined from

$$\begin{aligned} V_1(e_1^{m-}, e_2^{m-}, t_m^-) &= \max_{e_1'} V_1(e_1', e_2^{m-}, t_m^+)\big|_{break\ ties:\ e_1^{m-}} \\ &= V_1(e_1^{m+}, e_2^{m-}, t_m^+) \quad (50)\\ e_1^{m+} &= \operatorname*{argmax}_{e_1'} V_1(e_1', e_2^{m-}, t_m^+)\big|_{break\ ties:\ e_1^{m+}=e_1^{m-}} \ . \quad (51)\end{aligned}$$

We remind the reader that we break ties by staying at the current level (if that is a maxima of equation (51)) or preferring the lowest emission level (if the current state is not a maxima). Consequently, $e_1^{m+} = e_1^{m+}(e_1^{m-}, e_2^{m-}, t_m^+)$ since dependence on e_1^{m-} is induced by the tie-breaking rule.

At time t_{m+1}, player two chooses a control, with player one's control fixed at $e_1^{(m+1)-}$,

$$\begin{aligned} V_1(e_1^{(m+1)-}, e_2^{(m+1)-}, t_{m+1}^-) &= V_1(e_1^{(m+1)-}, e_2^{(m+1)+}, t_{m+1}^+) \quad (52)\\ V_2(e_1^{(m+1)-}, e_2^{(m+1)-}, t_{m+1}^-) &= V_2(e_1^{(m+1)-}, e_2^{(m+1)+}, t_{m+1}^+) \ . \quad (53)\end{aligned}$$

Player two's control is determined from

$$\begin{aligned} V_2(e_1^{(m+1)-}, e_2^{(m+1)-}, t_{m+1}^-) &= V_2(e_1^{(m+1)-}, e_2^{(m+1)+}, t_{m+1}^+) \\ &= \max_{e_2'} V_2(e_1^{(m+1)-}, e_2', t_{m+1}^+)\big|_{break\ ties:\ e_2^{(m+1)-}} \quad (54)\\ e_2^{(m+1)+} &= \operatorname*{argmax}_{e_2'} V_2(e_1^{(m+1)-}, e_2', t_{m+1}^+)\big|_{break\ ties:\ e_2^{(m+1)+}=e_2^{(m+1)-}} \\ &= R_2(e_1^{(m+1)-}; e_2^{(m+1)-}; t_{m+1}^+) \ , \quad (55)\end{aligned}$$

where $R_2(e_1^{(m+1)-}; e_2^{(m+1)-}; t_{m+1}^+)$ is the best response function of player two to player one's control $e_1^{(m+1)-}$. Note that the tie-breaking strategy induces a dependence on the state $e_2^{(m+1)-}$ in $R_2(\cdot)$.

More generally, we can define player two's response function for arbitrary arguments $(\omega_1; \omega_2)$

$$R_2(\omega_1; \omega_2; t_{m+1}^+) \quad = \quad \operatorname*{argmax}_{\omega_2'} V_2(\omega_1, \omega_2', t_{m+1}^+)\big|_{break\ ties:\ R_2=\omega_2} \ . \tag{56}$$

Now, consider the limit where $t_{m+1} \to t_m$, so that

$$e_1^{(m+1)-} \to e_1^{m+} \ ; \ e_2^{(m+1)-} \to e_2^{m-} \ ; \ t_{m+1}^- \to t_m^+ \ . \tag{57}$$

Using equation (57) in equation (52) gives

$$V_1(e_1^{m+}, e_2^{m-}, t_m^+) \quad = \quad V_1(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \ , \tag{58}$$

while equation (57) in equations (54-55) gives

$$V_2(e_1^{m+}, e_2^{m-}, t_m^+) \quad = \quad V_2(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \tag{59}$$

$$e_2^{(m+1)+} \quad = \quad R_2(e_1^{m+}; e_2^{m-}; t_{m+1}^+) \ . \tag{60}$$

From equations (58) and (60) we have

$$V_1(e_1^{m+}, e_2^{m-}, t_m^+) = V_1(e_1^{m+}, R_2(e_1^{m+}; e_2^{m-}; t_{m+1}^+), t_{m+1}^+) \ , \tag{61}$$

and replacing e_1^{m+} by e_1' in equation (61) gives

$$V_1(e_1', e_2^{m-}, t_m^+) = V_1(e_1', R_2(e_1'; e_2^{m-}; t_{m+1}^+), t_{m+1}^+) \ . \tag{62}$$

Recall that (from equation (50))

$$V_1(e_1^{m-}, e_2^{m-}, t_m^-) \;=\; \max_{e_1'} V_1(e_1', e_2^{m-}, t_m^+)\big|_{break\ ties:\ e_1^{m-}} \;, \tag{63}$$

so that substituting equation (62) into equation (63) gives

$$\begin{aligned} V_1(e_1^{m-}, e_2^{m-}, t_m^-) &= \max_{e_1\prime} V_1(e_1', R_2(e_1'; e_2^{m-}; t_{m+1}^+), t_{m+1}^+)\big|_{break\ ties:\ e_1^{m-}} \\ &= V_1(e_1^{m+}, R_2(e_1^{m+}; e_2^{m-}; t_{m+1}^+), t_{m+1}^+) \\ e_1^{m+} &= \operatorname*{argmax}_{e_1'} V_1(e_1', R_2(e_1'; e_2^{m-}; t_{m+1}^+), t_{m+1}^+)\big|_{break\ ties:\ e_1^{m-}} \;. \end{aligned} \tag{64}$$

From equations (49) and (59-60) we also have that

$$\begin{aligned} V_2(e_1^{m-}, e_2^{m-}, t_m^-) &= V_2(e_1^{m+}, e_2^{m-}, t_m^+) \\ &= V_2(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \\ e_2^{(m+1)+} &= R_2(e_1^{m+}; e_2^{m-}; t_{m+1}^+) \;. \end{aligned} \tag{65}$$

In summary, equations (64-65) give

$$\begin{aligned} V_1(e_1^{m-}, e_2^{m-}, t_m^-) &= V_1(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \\ V_2(e_1^{m-}, e_2^{m-}, t_m^-) &= V_2(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \\ & \quad e_1^{m+} = \operatorname*{argmax}_{e_1'} V_1(e_1', R_2(e_1'; e_2^{m-}; t_{m+1}^+), t_{m+1}^+)\big|_{break\ ties:\ e_1^{m-}} \\ & \quad e_2^{(m+1)+} = R_2(e_1^{m+}; e_2^{m-}, t_{m+1}^+) \;, \end{aligned} \tag{66}$$

which, from Definition 3, we recognize as a Stackelberg game if $t_{m+1}^+ \to t_m^+$.

Proposition 3 follows immediately:

Proposition 3 (Limit of Interleaved game)**.** *Suppose we have an Interleaved game at times t_m, given by equations (48-55). Suppose $t_{m+1} - t_m = \Delta t$, and that player one makes decisions for m even, while player two acts optimally for m odd. Consider fixing player one's decision* *times* $t_{2i}, i = 0, 1, \ldots,$ *and moving player two decision times* $t_{2i+1}, i = 0, 1, \ldots,$ *such that*

$$\begin{aligned} &(t_{2i+1} - t_{2i}) \to 0^+ \ ; \ i = 0, 1, 2, \ldots \\ &t_{2i} - t_{2(i-1)} = 2\Delta t \ ; \ i = 1, 2, \ldots \end{aligned} \tag{67}$$

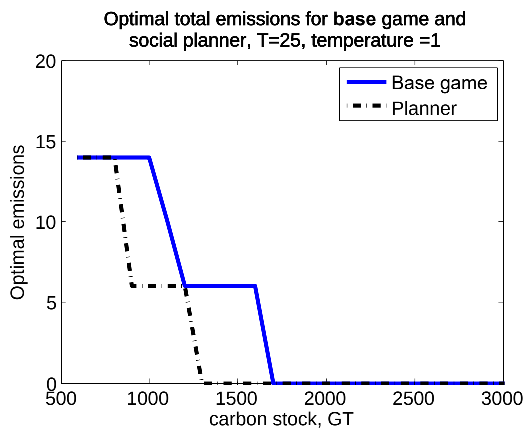
then the Interleaved game becomes a Stackelberg game.

E Additional results: Changing the terminal time

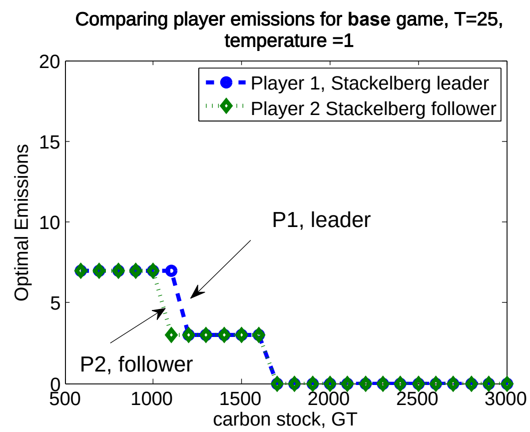
The terminal time for the analysis is set at 150 years. After 150 years it is assumed that due to a technological breakthrough, emissions no longer contribute to the stock of carbon, but do add benefits. We could imagine any carbon produced by burning fossil fuels is immediately captured and stored. At the boundary $t = T$ the temperature is set to the long run mean implied by the particular stock of carbon given by the state variable S. Utility at the boundary is set to be the present value of an infinite stream of utility from emissions (now harmless) set to their maximum level, and temperature remaining at the long run mean. This is an arbitrary assumption. The logic is that even with a technological breakthrough the earth will be left to bear the consequences of past carbon emissions for a long time to come. As a check on the results we ran cases with $T = 25$ and $T = 300$.

Figure 8 compares the optimal controls for $T = 150$ (lower two diagrams) with $T = 25$ (upper two diagrams) for the base Stackelberg game and the social planner. We observe that in the $T = 25$ case, the optimal controls are cut back at a lower carbon stock than when $T = 150$. This makes sense as with $T = 25$ there is much less time to react and have an impact on the final stock of carbon, and hence the terminal value of the temperature.

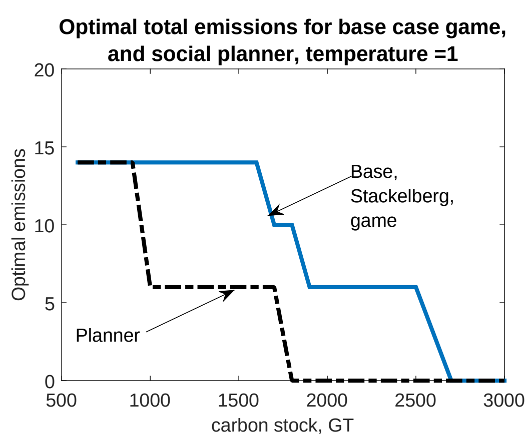
Optimal emissions for $T = 300$ versus $T = 150$ were also compared. These two cases are very similar, indicating that utility beyond 150 years is not having a large impact on results.



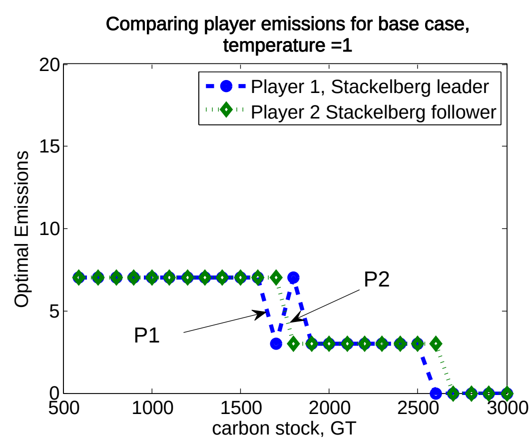
(a) Base game and Social Planner, T=25



(b) Base game individual players, T=25



(c) Base game and Social Planner, T=150



(d) Base game individual players, T=150

Figure 8: Comparing optimal controls for different terminal times, base Stackelberg game and the Social Planner, time = 0. State variables $e1 = e2 = 10$Gt. Temperature is at 1°C above preindustrial levels. P1 refers to player 1, P2 refers to player 2.

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