Chapter 7

Review Question Answers

7.1 Chapter 2

Question 1:

a) There are multiple $C(\cdot)$ which satisfy the Weak Axiom. Note, however, that you have to check back and forth to make sure that the WA is indeed satisfied. (I.e., $C(\{x,y,z\}) = \{x\}$, $C(\{x,y\}) = \{x,y\}$ does not satisfy the axiom since while the check for x seems to be ok, you also have to check for y, and there it fails.) One choice structure that does work is $C(\{x,y,z\}) = \{x\}, C(\{x,z,w\}) = \{x\}, C(\{y,w,z\}) = \{w\}, C(\{y,w\}) = \{w\}, C(\{x,z\}) = \{x\}, C(\{x,w\}) = \{x\}, C(\{x\}) = \{x\}$.

b) Yes (I thought of that first, actually, in deriving the above) it is $x \succeq w \succeq y \succeq z$.

c) Yes, it is transitive.

d) I was aiming for an application of out Theorem: our set of budget sets $\mathcal{B}$ does not contain all 2 and 3 element subsets of X. Missing are $\{x,y,w\}, \{x,y\}, \{y,z\}, \{w,z\}$.

e) The best way to go about this one is to determine where we can possibly get this to work. Examination of the sets B shows that the two choices y, x only appear in one of the sets and thus must be our key if we want to satisfy the WA without having rational preferences. Some fiddling reveals that the following works: $C(\{x,y,z\}) = \{x\}, C(\{x,z,w\}) = \{w\}, C(\{y,w,z\}) = \{y\}, C(\{y,w\}) = \{y\}, C(\{x,z\}) = \{x\}, C(\{x,w\}) = \{w\}, C(\{x\}) = \{x\}$. The problem is intransitivity, since the above implies that $y \succeq w \succeq x \succeq z$ but we also have $x \succeq y$!

Question 2: Here you have to make sure to maximize income for any given

work hrs	Part a 1 then 2	2 then 1	Max	Partb 1 then 2	2 then 1	Max
1	112	108	112	112	108	112
2	124	116	124	124	116	124
3	136	130	136	136	$130\frac{2}{3}$	136
4	148	144	148	148	$145\frac{1}{3}$	148
5	160	158	160	160	160	160
6	172	172	172	172	$174\frac{2}{3}$	$174\frac{2}{3}$
7	188	186	188	188	$189\frac{1}{3}$	$189\frac{1}{3}$
8	204	200	204	204	204	204
9	212	212	212	212	216	216
10	220	224	224	220	228	228
11	234	236	236	$234\frac{2}{3}$	240	240
12	248	248	248	$249\frac{1}{3}$	252	252
13	262	260	262	264	264	264
14	276	272	276	$278\frac{2}{3}$	276	$278\frac{2}{3}$
15	290	288	290	$293\frac{1}{3}$	292	$293\frac{1}{3}$
16	304	304	304	308	308	308

Table 7.1: Table 1: Computing maximal income

amount of work. In parts (a) and (b) you have to choose to work either job 1 then job 2 (after 8 hours in job 1) or job 2 then job 1. Simply plotting the two and then taking the outer hull (i.e., the highest frontier) for each leisure level gives you the frontier. In (a) they only cross twice (at 9 and 12 hours of work) while in part (b) they cross 4 times. You can best see this effect by considering a table in which you tabulate total hours worked against total income, computed by doing job 1 first, and by doing job 2 first. This is shown in Table 1. In neither part a) nor in part b) is the budget set convex.

c) This is a possibly quite involved problem. The intuitive answer is that it will not matter since marginal and average pay is (weakly) increasing in both jobs. Here is a more general treatment of these questions:
We really are faced with an maximization problem, to max income given the constraints, for any given total amount worked. Let h_1 and h_2 denote hours worked in jobs 1 and 2, respectively. Then the objective function is $I(h_1, h_2) = h_1 w_1(h_1) + h_2 w_2(h_2)$, where $w_i(h_i)$ are the wage schedules. The wage schedules have the general form $w_1(h_1) = \begin{cases} \underline{w}_1 & \text{if } h_1 \leq C_1 \\ \overline{w}_1 & \text{if } h_1 \geq C_1 \end{cases}$ and $w_2(h_2) = \begin{cases} \underline{w}_2 & \text{if } h_2 \leq C_2 \\ \overline{w}_2 & \text{if } h_2 \geq C_2 \end{cases}$, where $\underline{w}_i < \overline{w}_i$. I ignore here that no hours above 8 are possible for either job, choosing to put that information into the constraints later.

Consider now the iso-income curves in (h_1, h_2) space which result. We will have four regions to consider, namely $A = \{(h_1, h_2)|h_1 \leq C_1, h_2 \leq C_2\}$, $B = \{(h_1, h_2)|h_1 \leq C_1, h_2 \geq C_2\}$, $C = \{(h_1, h_2)|h_1 \geq C_1, h_2 \leq C_2\}$, $D = \{(h_1, h_2)|h_1 \geq C_1, h_2 \geq C_2\}$. The slope of the iso-income curves for the regions is easily seen to be the negative of the ratio of wages, so we have $S(A) = -\underline{w}_1/\underline{w}_2$, $S(B) = -\underline{w}_1/\overline{w}_2$, $S(C) = -\overline{w}_1/\underline{w}_2$, $S(D) = -\overline{w}_1/\overline{w}_2$. It is obvious that $S(C) < S(D)$ and $S(A) < S(B)$, as well as that $S(C) < S(A)$ and $S(D) < S(B)$. This implies, of course, that $S(C) < S(A) < S(B)$ as well as that $S(C) < S(D) < S(B)$, with the comparison of $S(A)$ to $S(D)$ indeterminate. (But luckily not needed in any case.) The important fact which follows from all of this is that the iso-income curves are all concave to the origin and piece-wise linear.

Now superimpose the choice sets onto this. Note that without any restrictions $H = h_1 + h_2$, that is, for any given number of hours H the hours in each job are "perfect substitutes". These iso-hour curves are all straight lines with a slope of -1. (For our parameters all of $S(A), S(C), S(D)$ are less than -1, while $S(B) > -1$, but his is not important.) For parts (a) and (b) the feasible set consists of the boundaries of the 8×8 square of feasible hours, where either $h_i = 0, h_j < 8$, or where $h_i = 8, 0 \leq h_j \leq 8$. The choice set is thus given by the intersection of the iso-hour lines with the feasible set (the box boundary). In part (c) this restriction is removed and the whole interior of the box is feasible. Due to the concavity to the origin of the iso-income lines this is of no relevance, however. Note how I have used our usual techniques of iso-objective curves and constraint sets to approach this problem. Works pretty well, doesn't it!

d) Now we "clearly" take up jobs in decreasing order of pay, starting with the highest paid and progressing to the lower paid ones in order. The resulting budget set will be convex.

Question 3: The consumer will

$$\max_x \left\{x_1^{0.3}x_2^{0.6} + \lambda\left(m - x_1p_1 - x_2p_2\right)\right\}$$

which leads to the first order conditions

$$0.3x_1^{-0.7}x_2^{0.6} = \lambda p_1, \quad 0.6x_1^{0.3}x_2^{-0.4} = \lambda p_2, \quad x_1p_1 + x_2p_2 = m.$$

The utility function is quasi-concave (actually, strictly concave in this case) and the budget set convex, so the second order conditions will be satisfied. Combining the first two first order conditions we get

$$\frac{0.3x_2}{0.6x_1} = \frac{p_1}{p_2} \quad \Longrightarrow \quad x_2 = \frac{2p_1}{p_2}x_1.$$

Substitute into the budget constraint and simplify:

$$x_1 p_1 + \frac{2p_1}{p_2} x_1 p_2 = m \quad \Longrightarrow \quad x_1 = \frac{m}{3p_1}.$$

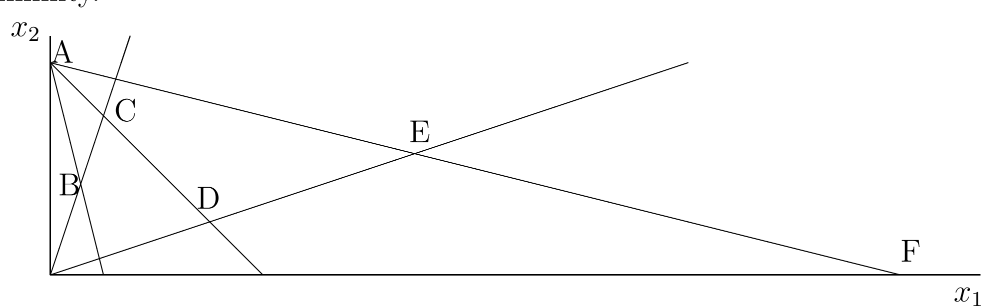
Now use this to solve for x_2: $x_2 = \frac{2m}{3p_2}$. So $(x_1(p,m), x_2(p,m)) = \left(\frac{m}{3p_1}, \frac{2m}{3p_2}\right)$.

To find the particular quantity demanded, simply plug in the numbers and simplify:

$$x_1 = \frac{3 \times 412 + 1 \times 72}{3 \times 3} = \frac{412 + 24}{3} = \frac{436}{3};$$

$$x_2 = \frac{2(3 \times 412 + 1 \times 72)}{3 \times 1} = \frac{2(412 + 24)}{1} = 872.$$

Question 4: The key is to realize that this utility function is piece-wise linear with line segments at slopes $-5, -1, -1/5$, from left to right. The segments join at rays from the origin with slopes 3 and 1/3. Properly speaking, neither the Hicksian nor the Marshallian demands are functions. The function has either a perfect substitute or Leontief character. In the former the substitution effects approach infinity, in the latter they are zero. Demands are easiest derived from the price offer curve, which is a nice zigzag line. It starts at the intercept of the budget with the vertical axis (point A). It follows the indifference curve segment with -5 slope to the ray with slope 3. Call this point B. From there it follows the ray with slope 3 until that ray intersects a budget drawn from A with a slope of 1 (point C). It then continues on this budget and the coinciding indifference curve segment to the ray with slope 1/3 (point D). Up along that ray to an intersection with a budget from A with slope 1/5 (point E), along that budget to the intercept with the horizontal axis (point F), and then along the horizontal axis off to infinity.



Now we can solve for the demands along the different pieces of the offer curve and get the Marshallian demand. Note that demand is either a whole range, or a "proper demand". The ranges can be computed from the endpoints (i.e., A to B, C to D, E to F.) Along the rays demand is solved as for a Leontief consumer: we know the ratio of consumption, we know the budget. So for example on the first ray segment (B to C) we know that $x_2 = 3x_1$. Also, $p_1x_1 + p_2x_2 = w$. Hence $x_1(p_1 + 3p_2) = w$, and $x_1 = w/(p_1 + 3p_2)$. (For the Hicksian demand we simply need to fix one indifference curve and compute the points along it. We then get either a segment (like A to B above), or we stay at a kink for a range of prices.) The demands for good 1 therefore are

$$x_1(p, w) = \begin{cases} 0, & \text{if } p_1/p_2 > 5; \\ [0, 5w/(8p_1)], & \text{if } 5 = p_1/p_2; \\ w/(3p_2 + p_1), & \text{if } 5 > p_1/p_2 > 1; \\ [w/(4p_1), 3w/(4p_1)], & \text{if } p_1/p_2 = 1; \\ 3w/(3p_1 + p_2), & \text{if } 1 > p_1/p_2 > 1/5; \\ [3w/(8p_1), w/p_1], & \text{if } p_1/p_2 = 1/5; \\ w/p_1, & \text{if } 1/5 > p_1/p_2. \end{cases}$$

$$h_1(p, u) = \begin{cases} 0, & \text{if } p_1/p_2 > 5; \\ [0, u/8], & \text{if } p_1/p_2 = 5; \\ u/8, & \text{if } 5 > p_1/p_2 > 1; \\ [u/8, 3u/16], & \text{if } p_1/p_2 = 1; \\ 3u/16, & \text{if } 1 > p_1/p_2 > 1/5; \\ [3u/16, u], & \text{if } p_1/p_2 = 1/5; \\ u, & \text{if } 1/5 > p_1/p_2. \end{cases}$$

The demands for good 2 are similar and left as exercise. The income expansion paths and Engel curves can be whole regions at price ratios 1,5,1/5, otherwise the income expansion paths are the axes or rays, and the Engel curves are straight increasing lines.

Question 5: The elasticity of substitution measures by how much the consumption ratio changes as the price ratio changes (both measured in percentages.) In other words, as the price ratio changes the slope of the budget changes and we know this will cause a change in the ratio of the quantity demanded of the goods. But by how much? The higher the value of the elasticity, the larger the response in demands.

Question 6: First we need to realize that the utility index which each function assigns to a given consumption point does not have to be the same. Instead, as long as the MRS is identical at every point, two utility functions represent the same preferences. So instead of taking the limit of the utility function directly, we will take the limits of the MRS and compare those to the MRSs of the other functions.

$$MRS = \frac{u_1}{u_2} = \frac{(1/\rho)(x_1^\rho + x_2^\rho)^{(1-\rho)/\rho}(\rho x_1^{\rho-1})}{(1/\rho)(x_1^\rho + x_2^\rho)^{(1-\rho)/\rho}(\rho x_2^{\rho-1})} = \frac{x_1^{\rho-1}}{x_2^{\rho-1}} = \frac{x_2^{1-\rho}}{x_1^{1-\rho}}.$$

The MRSs for the other functions are

$$\text{CD: } \frac{x_2}{x_1}; \quad \text{Perfect Sub: } 1; \quad \text{Leon: } 0, \text{ or } \infty.$$

So, consider the Leontief function $min\{x_1, x_2\}$. Its MRS is 0 or ∞. But as $\rho \to -\infty$ we see that $(x_2/x_1)^{1-\rho} \to (x_2/x_1)^{\infty}$. But if $x_2 > x_1$ the fraction is greater than 1 and an infinite power goes to infinity. If $x_2 < x_1$ the fraction is less than one and the power goes to zero. The Cobb-Douglas function $x_1 x_2$ has MRS x_2/x_1. But as $\rho \to 0$ the MRS of our function is just that. The perfect substitute function $x_1 + x_2$ has a constant MRS of 1. But as $\rho \to 1$ the MRS of our function is $(x_2/x_1)^0 = 1$. Therefore the CES function "looks like" those three functions for those choices of ρ. The parameter ρ essentially controls the curvature of the IC's.

Question 7: Set up the consumer's optimization problem:

$$\max_{x_1,x_2,x_3} \{x_1 + \ln x_2 + 2\ln x_3 + \lambda(m - p_1 x_1 - p_2 x_2 - p_3 x_3)\}.$$

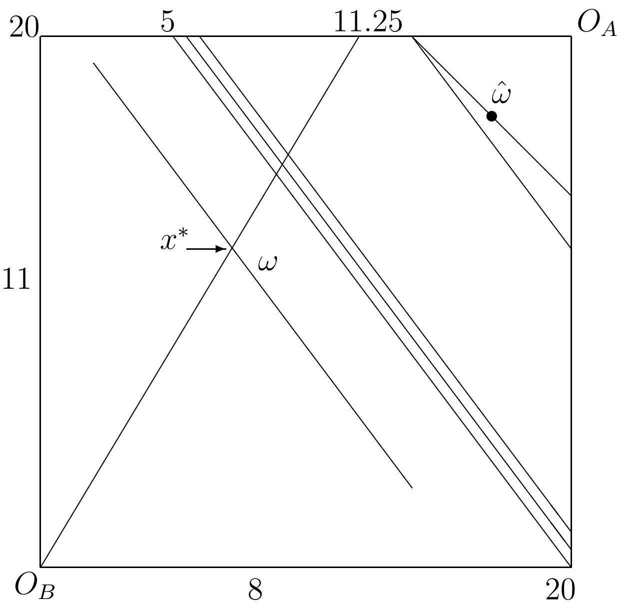
The FOCs are

$$1 - \lambda p_1 = 0; \quad \frac{1}{x_2} - \lambda p_2 = 0; \quad \frac{2}{x_3} - \lambda p_3 = 0$$

and the budget. The first of these allows us to solve for $\lambda = 1/p_1$. Therefore the second and third give us $x_2 = p_1/p_2$ and $x_3 = 2p_1/p_3$. Combining this with the budget we get $x_1 = m/p_1 - 3$. Of course, this is only sensible if $m > 3p_1$. If it is not we must be at a corner solution. In that case $x_1 = 0$ and all money is spent on x_2 and x_3. The second and third FOC above tell us that $x_3/x_2 = 2p_2/p_3$. Hence (remember $x_1 = 0$ now) $m = p_2 x_2 + 2p_2 x_2$ and $x_2 = m/(3p_2)$ while $x_3 = 2m/(3p_3)$. So we get

$$x(p,m) = \begin{cases} \left(\frac{m}{p_1} - 3, \frac{p_1}{p_2}, 2\frac{p_1}{p_3}\right) & \text{if } m > 3p_1 \\ \left(0, \frac{m}{3p_2}, \frac{2m}{3p_3}\right) & \text{if } m \leq 3p_1 \end{cases}$$

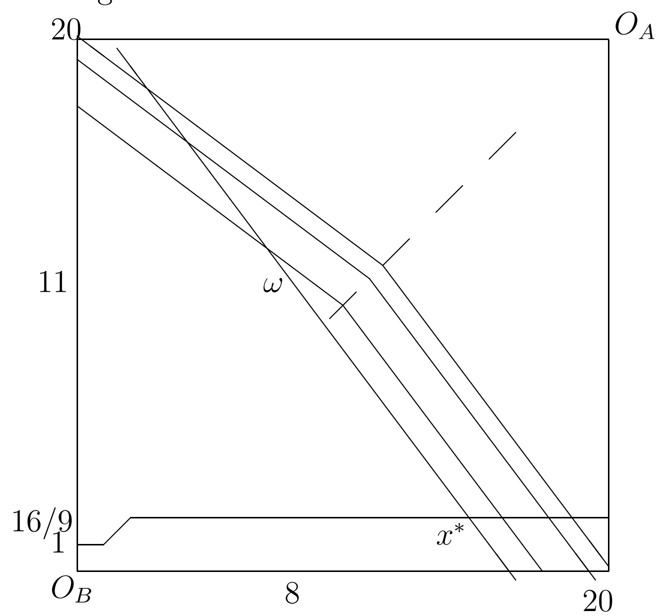
Question 8: Here we have a pure exchange economy with 2 goods and 2 consumers. We can best represent this in an Edgeworth box.



Suppose x_1 is on the horizontal axis and x_2 is on the vertical, and let consumer B have the lower left hand corner as origin, consumer A the upper right hand corner. (I made this choice because I like to have the "harder" consumer oriented the usual way.) The dimensions of the box are 20 by 20 units. The first thing to do is to find the Pareto Set (the contract curve), since we know that any equilibrium has to be Pareto efficient. The MRS for person A is 4/3, the MRS for person B is $3x_2/(4x_1)$. Therefore the Pareto Set is defined by $x_2/x_1 = 16/9$ (in person B's coordinates.) This is a straight ray from B's origin with a slope greater than 1, and therefore above the main diagonal. The Pareto set is this ray and the portion of the upper boundary of the box from the ray's intersection point to the origin of A. There now are 2 possibilities for the equilibrium. Either it is on the ray, and therefore must have a price ratio of 4/3. Or it is on the upper boundary of the box, in which case the price ratio must be below 4/3, but we know that B's consumption level for good 2 is 20. In the first case we have 2 equations defining equilibrium. The ray, $x_2 = 16x_1/9$, and the budget line $(x_2 - 11) = 4(8 - x_1)/3$. From this we get $16x_1/9 - 11 = 32/3 - 12x_1/9$ and from that $28x_1/9 = 65/3$ and thus $x_1 = 195/28 < 20$. It follows that $x_2 = (16 \times 195)/(9 \times 28) = 780/63 = 260/21 < 20$. Since both of B's consumption points are strictly within the interior of the box, we are done. All that remains is to compute A's allocation. The equilibrium is therefore

$$(p*, (x^A), (x^B)) = \left(\frac{4}{3}, \left(20 - \frac{195}{28}, 20 - \frac{780}{63}\right), \left(\frac{195}{28}, \frac{780}{63}\right)\right).$$

Question 9: Again we have a square Edgeworth box, 20×20. Again I choose to put consumer B on the bottom left origin. B's preferences are quasi-linear with respect to x_1, A's are piece-wise linear with slopes 4/3 and 3/4 which meet at the kink line $x_2 = x_1$ (in A's coordinates!) which coincides with the main diagonal. The MRS for B's preferences is $3x_2/4$. We have Pareto optimality when $3x_2/4 = 4/3 \rightarrow x_2 = 16/9$ and when $3x_2/4 = 3/4 \rightarrow x_2 = 1$. So, the Pareto Set is the vertical axis from B's origin to $x_2^B = 1$, the horizontal line $x_2^B = 1$ to the main diagonal (the point $(1, 1)$ in other words), up the main diagonal to the point $(x_1^B, x_2^B) = (16/9, 16/9)$, from there along the horizontal line $x_2^B = 16/9$ to the right hand edge of the Box, and then up that border to A's origin.



By inspection, the most likely candidate for an equilibrium is a price ratio of 4/3 with an allocation on the second horizontal line segment. Let us attempt to solve for it. First, the budget equation (in B's coordinates) is $3(x_2 - 11) = 4(8 - x_1)$. Second, we are presuming that $x_2 = 16/9$. So we get $16/3 - 33 = 32 - 4x_1$, or $x_1 = 14\frac{11}{12}$. Since this is less than 20 we have found an interior point and are done. The equilibrium is

$$(p*, (x^A), (x^B)) = \left(\frac{4}{3}, \left(5\frac{1}{12}, \frac{164}{9}\right), \left(14\frac{11}{12}, \frac{16}{9}\right)\right).$$

Question 10: To prove this we need to show the implication in both directions: $(\Leftarrow)$: Suppose $x \succ y$. Then $\exists B,\ x, y \in B$ with the property that $x \in C(B),\ y \notin C(B)$. Consider all other $B' \in \mathcal{B}$ with the property that $x, y \in B'$. By the Weak Axiom $\nexists B'$ with $y \in C(B')$ since otherwise the set B would violate the weak axiom (applied to the choice y with the initial set B'.) Therefore $x \succ^* y$.
$(\Rightarrow)$: Let $x \succ^* y$. The first part of the definition requires $\exists B,\ x, y, \in B,\ x \in C(B)$. By the weak axiom there are two possibilities: either all $B' \in \mathcal{B}$ with $x, y \in B'$ have $\{x, y\} \in C(B')$ or none have $y \in C(B')$. The second part of

the definition requires us to be in the second case, but then $y \notin C(B)$, and so $x \succ y$.
If the WA fails a counter example suffices: Let $X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$, $C(\{x, y\}) = \{x\}$, $C(\{x, y, z\}) = \{x, y\}$. This violates the WA. $C(\{x, y\}) = \{x\}$ demonstrates that $x \succ y$ by definition. On the other hand it is not true that $x \succ^* y$ (let $B = \{x, y\}$ and $B' = \{x, y, z\}$ in the definition of $\succ^*$.

Question 11:

a) This is another 20 by 20 box, with the endowment in the centre. Suppose B's origin on the bottom left, A's the top right. As in question 8, A's indifference curves have a constant MRS of α and are perfect substitute type. B's ICs have a MRS of $\beta x_2/x_1$ and are Cobb-Douglas. The contract curve in the interior must have the MRSs equated, so it occurs where (in B's coordinates) $x_2/x_1 = \alpha/\beta$. This is a straight ray from B's origin and depending on the values of α and β it lies above or below the main diagonal. Since these cases are (sort of) symmetric we pick one, and assume that $\alpha/\beta > 1$. The contract curve is this ray and then the part of the upper edge of the box to A's origin.
As in question 8 there are two cases for the competitive equilibrium. Either it occurs on the part of the Contract curve interior to the box, or it occurs on the boundary of the box. In the first case the slope of the budget and hence the equilibrium price must be α, since both MRSs have that slope along the ray and in equilibrium the price must equal the MRS. Note that the budget now coincides with A's indifference curve through the endowment point. The equilibrium allocation is determined by the intersection of the contract curve and this budget/IC. So we have two equations in two unknowns:

$$\alpha = \frac{x_2 - 10}{10 - x_1} \quad \text{and} \quad x_2 = \frac{\alpha}{\beta} x_1.$$

Hence $\alpha(10 - x_1) = \alpha x_1/\beta$ or $\alpha\beta 10 = x_1(\alpha + \alpha\beta)$ and thus $x_1 = \beta 10/(1+\beta)$ and $x_2 = \alpha 10/(1 + \beta)$. These are the consumption levels for B. A gets the rest. The equilibrium thus would be $(p^*, (x_1^A, x_2^B), (x_1^B, x_2^B)) =$

$$\left(\alpha, \left(10\frac{2+\beta}{1+\beta}, 10\frac{2(1+\beta)-\alpha}{1+\beta}\right), \left(\frac{\beta 10}{1+\beta}, \frac{\alpha 10}{1+\beta}\right)\right)$$

which only makes sense if the allocation indeed is interior, that is, as long as $10\alpha/(1 + \beta) < 20$, or $(\alpha - \beta) < (2 + \beta)$.
If that is not true we find ourselves in the other case. In that case we know that we are looking for an equilibrium on the upper boundary of the box and thus know that $x_2^B = 20$ while $x_2^A = 0$. It remains to determine p and the allocations for good 1. At the equilibrium point the budget must be flatter

than A's IC (so that A chooses to only consume good 1). The allocation must also be the optimal choice for B and hence the budget must be tangent to B's IC, since for B this is an interior consumption bundle (interior to B's consumption set, that is.) So we again have to solve two equations in 2 unknowns:

$$\frac{\beta c_2}{c_1} = p \quad \text{and} \quad p = \frac{c_2 - 10}{10 - c_1} \quad \text{while} \quad c_2 = 20.$$

It follows that $\beta 20(10 - c_1) = 10c_1$ and therefore $c_1 = 20\beta/(1 + 2\beta)$. A gets the rest. The equilibrium is therefore

$$(p^*, (x_1^A, x_2^A), (x_1^B, x_2^B)) = \left(1 + 2\beta, \left(20\frac{1+\beta}{1+2\beta}, 0\right), \left(\frac{\beta 20}{1+2\beta}, 20\right)\right).$$

b) All endowments above and to the right of the line $x_2 = 40 - 2x_1$ in B's coordinates will lead to a boundary equilibrium. All those on this line and below will lead to an interior equilibrium with $p = 2$.

Question 12:

a) The social planner's problem is

$$\max_l \left\{ \ln(4\sqrt{16 - l}) + \frac{1}{2}\ln(l) \right\}$$

which has first order condition

$$-\frac{1}{4\sqrt{16 - l}} \frac{2}{\sqrt{16 - l}} + \frac{1}{2l} = 0.$$

Hence $16 - l = l$ and so $l^* = 8,\ x^* = 8,\ c^* = 8\sqrt{2}$.

b) Since the consumer's problem requires profits, we solve for the firm first. $\max_x\{p4\sqrt{x} - wx\}$ has FOC $2p/\sqrt{x} = w$ and leads to firm labour demand of $x(p, w) = 4p^2/w^2$, consumption good supply of $c(p, w) = 8p/w$, and profits of $\pi(p, w) = 4p^2/w$.
The consumer will

$$\max_{c,l} \left\{ \ln c + \frac{1}{2}\ln l + \lambda\left(16w + \pi(p, w) - pc - wl\right) \right\}$$

which has first order conditions $1/c - \lambda p = 0$; $1/(2l) - \lambda w = 0$; $16w + \pi(p, w) = pc + wl$. The first 2 imply that $pc = 2lw$. Substituting into the third and using the profits computed above yields demand of $c(p, w) = 32w/(3p) + 8p/(3w)$ and leisure demand of $l(p, w) = 16/3 + 4p^2/(3w^2)$.
We can now solve for the equilibrium price ratio. Take any one market and set demand equal to supply. For the goods market this implies $32w/(3p) +$

$8p/(3w) = 8p/w$, and hence $p^2/w^2 = 2$. Substituting into the demands and supplies this gives $l^* = 8$, and hence all values are the same as in the social planner's problem in part a). You may want to verify that you could have solved for the price ratio from the labour market.
The complete statement of the general equilibrium is: The equilibrium price ratio is $p/w = \sqrt{2}$, the consumer's allocation is $(c, l) = (8\sqrt{2}, 8)$, and the firm produces $8\sqrt{2}$ units consumption good from 8 units input. Note that we cannot state profits without fixing one of the prices. So let $w = 1$ (so that we use labour as numeraire), then $p = \sqrt{2}$ and profits are 8.

7.2 Chapter 3

Question 1:

a) Zero arbitrage means that whichever way I move between periods, I get the same final answer. In particular I could lend in period 1 to collect in period three, or I could lend in period 1 to period 2, and then lend the proceeds to period 3. Hence the condition is

$$(1 + r_{12})(1 + r_{23}) = (1 + r_{13}).$$

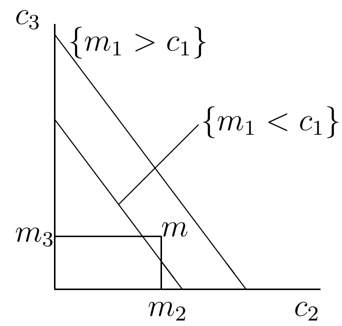
Note that if we where to treat r_{13} not as a simple interest rate but as a compounding one, we'd get $(1 + r_{12})(1 + r_{23}) = (1 + r_{13})^2$ instead.

b) You have to adopt one period as your viewpoint and then put all other values in terms of that period (by discounting or applying interest). With period 3 as the viewpoint I use period 3 future values for everything:

$$B = \{(c_1, c_2, c_3) | (1+r_{13})c_1 + (1+r_{23})c_2 + c_3 = (1+r_{13})m_1 + (1+r_{23})m_2 + m_3\}$$

Note that any other viewpoint is equally valid. The restriction in (a) means that it does not matter which interest rate I use to compute the forward value of c_1, say. Indeed, without that restriction I would get an infinite budget if it is possible to borrow infinite amounts. With some borrowing constraints in place I would have to compute the highest possible arbitrage profits for the various periods and compute the resulting budget.

c) This is a standard downward sloping budget line in (c_2, c_3) space with a slope of $-(1 + r_{23})$. It does not necessarily have to go through the endowment point (m_2, m_3), however. It will be below that point if $c_1 > m_1$ and above that point if $c_1 < m_1$.



Question 2:

a) The easy way to get this is to first ignore the technology. The market budget is a straight line with a slope of -1 through the point $(100, 100)$, which is truncated at the point $(160, 40)$, where the budget becomes vertical. Note that the gross rate of return is 1 since the interest rate is 0. Now consider the technology and the implications of zero arbitrage: Joe can move consumption from period 1 to period 2 in two ways, via the financial market, or via "planting". Both must yield the same gross rate of return at the optimum (why? we know that at the optimum of a maximization problem the last dollar allocated to each option must yield the same marginal benefit.) The gross rate of return at the margin is nothing but the marginal product of the technology, however. So, compute the MP ($5/\sqrt{x_1}$) and find the investment level at which the MP is 1.

$$\frac{5}{\sqrt{x_1}} = 1 \quad \longrightarrow \quad 5 = \sqrt{x_1} \quad \longrightarrow \quad x_1 = 25.$$

At optimal use at an interior optimum Joe invests 25 units (and collects 50 in the next period.) This means that from any point on the financial market budget Joe can move left 25 and up 50. So that gives a straight line with slope -1 which starts at $(0, 225)$ and goes to $(135, 90)$. After this point there is a corner solution in technology choice: Joe cannot use the market any more. The technology therefore may give a higher return than the market. So the budget follows the (flipped over to the left) technology frontier down to the point $(160, 40)$, and down to $(160, 0)$ from there.

b) First simplify the preferences (this step is not necessary!). Applying a natural logarithm gives the function $\hat{U}(c_1, c_2) = c_1^4 c_2^6$ which represents the same preferences. Applying the 10th root gives $\tilde{U}(c_1, c_2) = c_1^{.4} c_2^{.6}$ which also represents the same preferences and is recognized as a Cobb-Douglas. Now you can either compute the MRS ($2c_2/3c_1$) and set that equal to 1 (since most of the budget has a slope of -1 and we know that $MRS = Slope$ at the optimum.) That gives you two equations in two unknowns, and we can solve:

$$c_2 = 225 - c_1,\ c_2 = 3c_1/2 \quad \rightarrow \quad 450 = 5c_1 \quad \rightarrow \quad c_1 = 90,\ c_2 = 135.$$

We then double check that the assumption that we are on the -1 sloped portion of the budget was correct, which it is (by inspection.) Or you could use the demand function for C-D, so you know

$$(c_1, c_2) = \left(\frac{.4M}{p_1}, \frac{.6M}{p_2}\right) = \left(\frac{.4 \times 225}{1+0}, \frac{.6 \times 225}{1}\right) = (90, 135).$$

Now this is his final consumption bundle. In order to get there he invested 25 units, so on the "market budget" line he must have started at $(115, 85)$, and that required him to borrow 15 units.

In summary, he borrows 15, giving him 115, of which he invests 25, so he has 90 left to consume. In the next period he gets 100 from his endowment, 50 from the investment, for a total of 150, of which he has to use 15 to pay back the loan, so he can consume 135!

Question 3: I will not draw the diagram but describe it. You should refer to a rough diagram while reading these solutions to make sense of them.

a) The indifference curves have two segments with a slope of -1.3 and -1.2 respectively. The switch (kink) occurs where

$$23\left(\frac{12}{10}c_1 + c_2\right) = 22\left(\frac{13}{10}c_1 + c_2\right) \quad \rightarrow \quad c_2 = (22 \times 13 - 23 \times 12)c_1/10 = c_1.$$

b) Note that the budget has a slope of 1.25 which is less than 1.3 and more than 1.2, so she consumes at the kink. Thus she is on the kink line and the budget:

$$c_1 = c_2 \text{ and } -1.25 = \frac{c_2 - 8}{c_1 - 9} \quad \rightarrow \quad c_1 = c_2 = 77/9.$$

c) Again she consumes at the kink, so

$$c_1 = c_2 \text{ and } -1.25 = \frac{c_2 - 12}{c_1 - 5} \quad \rightarrow \quad c_1 = c_2 = 73/9.$$

d) Here we need to work back. Note that at the slopes implied by the interest rates she continues to consume at her kink line. The reason is that both 1.25 and 1.28 are bigger than 1.2, the slope of her lower segment, but less than 1.3, the slope of the steep segment. Hence optimal consumption is at the kink and she borrows if she has less period 1 endowment than period 2 endowment. She lends money if she has larger period 1 endowment than period 2 endowment. So for all endowments above the main diagonal she is a borrower, for all endowments below a lender.

e) Now she never trades. To lend money the budget slope is 1.18 which is less than either of her IC segment slopes. She would not want to lend ever at this rate no matter what her endowment. On the other hand, suppose she were to borrow. The budget slope is 1.32 which is steeper than even her steepest IC segment. She would not borrow. Thus she remains at the kink in her budget (the endowment point) no matter where it is.

Question 4: Again I will not draw the diagram but describe it. You should refer to a rough diagram while reading these solutions to make sense of them.

a) This is a 20 by 10 box. Suppose A's origin on the bottom left, B's the top right. A's indifference curves have a MRS of $c_2/(\alpha c_1)$ and are nice C-D type curves. B's ICs have a MRS of $1/\beta$ and are straight lines. The contract curve in the interior must have the MRSs equated (from Econ 301: for differentiable utility functions an interior Pareto optimum has a tangency), so it occurs where $c_2/c_1 = \alpha/\beta$. This is a straight ray from A's origin and depending on the values of α and β it lies above or below the main diagonal. Since these cases are (sort of) symmetric we pick one, and assume that $\alpha/\beta > 1/2$. The contract curve is this ray and then the part of the upper edge of the box to B's origin.

b) There are two cases, either the Contract curve ray is shallow enough that the equilibrium occurs on it, or it is so steep that the equilibrium occurs on the top boundary of the box. In the first case the slope of the budget and hence the equilibrium price must be $1/\beta$, since both MRSs have that slope along the ray and in equilibrium the price must equal the MRS. So the equilibrium interest rate is $(1-\beta)/\beta$. Note that the budget now coincides with player B's indifference curve through the endowment point. Hence the ray of the contract curve must intersect that, and it does so only if it intersects the top boundary to the right of the intersection of B's IC with the boundary. The latter occurs at $(4(3-\beta), 10)$. The former occurs at $(10\beta/\alpha, 10)$. So the interior solution obtains if $10\beta/\alpha > 4(3-\beta)$, or if $10\beta > 12\alpha - 4\alpha\beta$. In that case the equilibrium allocations are derived by solving the intersection of the budget and the ray:

$$c_2 = \alpha c_2/\beta \text{ and } -\frac{1}{\beta} = \frac{c_2 - 6}{c_1 - 12} \quad \rightarrow \quad c_1^A = \frac{6(2+\beta)}{1+\alpha}, c_2^A = \frac{\alpha 6(2+\beta)}{\beta(1+\alpha)}.$$

B gets the remainder.

In the other case, when the ray fails to intersect B's IC, we know that we are looking for an equilibrium on the upper boundary of the box (so $c_2^A = 10$ and $c_2^B = 0$.) At this point we must have a budget flatter than B's IC (so that B chooses to only consume good 1). It must also be tangent to

A's IC, since for player A this is an interior consumption bundle (interior to his consumption set, that is.) So we require $1+r = 10/(\alpha c_1)$ to have the tangency, and we require $1+r = (10-6)/(12-c_1)$ in order to be on the budget line. These are two equations in two unknowns again, so we solve: $c_1^A = 60/(5+2\alpha)$ and $r = (5-4\alpha)/(6\alpha)$. B gets the rest of good 1, of course.

7.3 Chapter 4

Question 1: We wish to show that for any concave $u(x)$

$$\frac{1}{3}u(24) + \frac{1}{3}u(20) + \frac{1}{3}u(16) \geq \frac{1}{2}u(24) + \frac{1}{2}u(16).$$

We can do the following: first bring the $u(24)$ and $u(16)$ to the RHS:

$$\frac{2}{6}u(20) \geq \frac{1}{6}u(24) + \frac{1}{6}u(16).$$

Then multiply both sides by 3:

$$u(20) \geq \frac{1}{2}u(24) + \frac{1}{2}u(16).$$

The LHS of this represents a certain outcome of 20, the RHS a lottery with 2 equally likely outcomes. Now note that

$$\frac{1}{2}24 + \frac{1}{2}16 = 20.$$

That is, the expected value of the lottery on the RHS of the last inequality above is equal to the expected value of the degenerate lottery on the LHS. Therefore this penultimate inequality must be true, since it coincides with the definition of a risk averse consumer. (utility of expectation greater than expectation of utility.)

Question 2: The certainty equivalent is defined by

$$U(CE) = \sum p_i u(x_i) = \int u(x)\mathrm{d}F(x).$$

Using the particular function we are given:

$$\sqrt{CE} = \alpha\sqrt{3600} + (1-\alpha)\sqrt{6400}$$

$$CE = (\alpha 60 + (1-\alpha)80)^2 = (80-20\alpha)^2.$$

Note that the expected value of the gamble is $\mathrm{E}(w) = \alpha 3600 + (1-\alpha)6400 = 6400 - \alpha 2800$ and thus the maximal fee this consumer would pay for access to fair insurance would be the difference $\mathrm{E}(w) - CE = 400\alpha(1-\alpha)$.

Question 3: The coefficient of absolute risk aversion is defined as $r_A = -u''(w)/u'(w)$. Computing this for both functions we get

$$u(w) = \ln w \longrightarrow r_A = \frac{1}{w}; \qquad u(w) = 2\sqrt{w} \longrightarrow r_A = \frac{1}{2w}.$$

Therefore the two consumers exhibit equal risk aversion if the second consumer has half the wealth of the first. Their relative risk aversion coefficients (defined as $-u''(w)w/u'(w)$ are 1 and 1/2, respectively. That means that while, if the logarithm consumer has twice the wealth as the root consumer, he will have the same attitude towards a fixed dollar amount gamble, he will be more risk averse with respect to a gamble over a given proportion of wealth. (Note that the two statements don't contradict one another: a $1 gamble represents half the percentage of wealth for a consumer with twice the wealth!)

Question 4: Here we need an Edgeworth Box diagram, which is a square, 15 units a side. Suppose we have consumer A on the bottom left origin (B then goes top right). Suppose also that we put state R on the horizontal. Note that the certainty line is the main diagonal of the box! This observation is crucial, since it means that there is no aggregate risk!

General equilibrium requires that demand is equal to supply for each good, but we can't find those here (not knowing the consumers' tastes), so it is not useful information. But we also know that in general equilibrium the price ratio must equal each consumer's MRS (since GE is Pareto optimal and that requires MRSs to be equalized, at least for interior allocations.) Note that the two MRSs here are

$$MRS_A = \frac{\pi u'_A(c_R^A)}{(1-\pi)u'_A(c_S^A)} \qquad MRS_B = \frac{\pi u'_B(c_R^B)}{(1-\pi)u'_B(c_S^B)}$$

On the certainty line (the main diagonal) $c_S^A = c_R^A$ and $c_S^B = c_R^B$, so $MRS_A = MRS_B = \pi/(1-\pi)$. In other words, the certainty line for each consumer coincides and together they are the set of Pareto optimal points.

Hence the equilibrium price ratio must be $p^* = \pi/(1-\pi)$.

The allocation is now easily computed: we know the price ratio and the endowment, hence the budget line for the consumers. We also know that consumption is equal in both states. So

$$\frac{\pi}{1-\pi} = \frac{c_S^A - 5}{10 - c_R^A} = \frac{c^A - 5}{10 - c^A} \to c_S^A = c_R^A = 5(1+\pi)$$

and since $c_i^B = 15 - c_i^A$ we get $c_S^B = c_R^B = 5(2-\pi)$.

Question 5: **a)** $\max_x \{0.5u(10000(1+0.8x)) + 0.5u(10000(1.4-0.8x))\}$

b) The FOC for this is

$$0.5 \times 0.8u'(100000(1+0.8x)) - 0.5 \times 0.8u'(10000(1.4-0.8x)) = 0$$

$$\begin{aligned} \text{implies :} \quad & u'(10000(1+0.8x)) = u'(10000(1.4-0.8x)) \\ \text{implies :} \quad & 10000(1+0.8x) = 10000(1.4-0.8x) \end{aligned}$$

since she is risk averse. It follows that $1 + .8x = 1.4 - .8x$, and therefore that $1.6x = 0.4$, so that $x = 0.25$. One quarter, or 25% are invested in gene technology.

Question 6:

i) Denote the probability with which a ticket wins by π and the prize by P. A fair price for this lottery ticket would have to be a fraction p per dollar of prize such that $\pi(P - pP) - (1-\pi)pP = 0$, or $p = \pi$. Let us start with this as a benchmark case (we know that normally such a lottery would not be accepted.) Utility maximization requires that for a gambling consumer $v(w_0) \leq \pi v(w_0 + (1-p)P) + (1-\pi)v(w_0 - pP) + \mu_i$. Thus all consumers for whom $\mu_i \geq v(w_0) - \pi v(w_0 + (1-p)P) - (1-\pi)v(w_0 - pP)$ purchase a ticket. At a fair gamble this is

$$\begin{aligned} \mu_i &\geq v(w_0) - \pi v(w_0 + (1-\pi)P) - (1-\pi)v(w_0 - \pi P) \\ &> v(w_0) - v(\pi(w_0 + (1-\pi)P) + (1-\pi)(w_0 - \pi P)) \\ &= v(w_0) - v(w_0) \end{aligned}$$

(the second strict inequality follows from the definition of risk aversion). Clearly a strictly positive μ is required. Can the government make money on this? Well, assume that the price p above is fair ($p = \pi$) and let there be an additional charge of q. Now all consumers gamble for whom $\mu_i \geq v(w_0) - \pi v(w_0 + (1-\pi)P - q) - (1-\pi)v(w_0 - \pi P - q)$. While such a μ_i is larger than before, it exists (for small q in any case) as long as things are sufficiently smooth and the μ_i go that high. Note that those who gamble have a high utility for it (a high taste parameter μ_i) in this setting. Note that this implies that even though they lose money on average they have a higher welfare. (The anti-gambling arguments in public policy debates therefore come in two flavours: (i) your gambling is against my (religious) beliefs, and thus it ought to be banned, (ii) there are externalities: your lost money is really not yours but should have bought a lunch for your child/spouse/dog. Since your child/spouse/dog can't make you stop, we will on their behalf.)

ii) Now μ is fixed. Of course, the decision to gamble will still depend on the same inequality, namely

$$\mu > v(w_0) - \pi v(w_0 + (1-\pi)P - q) - (1-\pi)v(w_0 - \pi P - q).$$

We thus can translate this question into the question of how the right hand side depends on w_0 and how this dependency relates to the different behaviours of risk aversion with wealth. So, is the right hand side increasing or decreasing with wealth, and is this a monotonic relationship? The right hand side is related, of course, to the utility loss from going to the expected utility from the expected value (ignoring q for a minute.) Intuitively, we would expect the difference to be declining in wealth for constant absolute risk aversion: Constant absolute risk aversion implies a constant difference between the expected value and the certainty equivalent.[1] Let this difference be the base of a right triangle. Orthogonal to that we have the side which is the required distance between the two utilities. The third side must have a declining slope as wealth increases since it is related to the marginal utility of wealth at the certainty equivalent, which is declining in wealth by assumption. There you go, I'd expect the utility difference must fall with wealth.

More formally, consider the original inequality again and approximate the RHS by its second order Taylor series expansion (that way we get first and second derivatives, which we want in order to form r_A:

$$\begin{aligned}
&v(w_0) - \pi v(w_0^{\oplus}) - (1-\pi)v(w_0 - \pi P - q) \approx \\
&\quad v(w_0) - \pi(v(w_0) + \oplus v'(w_0) + \\
&\qquad \oplus^2 v''(w_0)/2) - (1-\pi)(v(w_0) - \ominus v'(w_0) + \ominus^2 v''(w_0)/2) \\
&\quad = -\pi \oplus v'(w_0) - \pi \oplus^2 v''(w_0)/2(1-\pi)(\ominus v'(w_0) - \ominus^2 v''(w_0)/2) \\
&\quad = v'(w_0)\left[(1-\pi)\ominus(1-\ominus r_A/2) - \pi \oplus (1-\oplus r_A/2)\right].
\end{aligned}$$

This looks more like it! Now note that we use $\ominus$ and $\oplus$ as positive quantities (which are not equal: $\ominus$ is larger!) Furthermore we know that (a) this quantity must be positive and (b) that π is probably a very small number. Now, if r_A is constant then the term in brackets is constant, but of course

[1] Is there a general proof for that? Note that constant r_A has for example the functional form $u(w) = -e^{-aw}$, for which the above is certainly true.

$v'(w)$ falls with w and thus the right hand side of our initial inequality (way above) falls. Any given μ is therefore more likely to be larger than it. Thus rich consumers participate, poor consumers don't if we have constant absolute risk aversion. If we have decreasing absolute risk aversion this effect is strengthened. Now, since relative risk aversion is just $r_A w$, it follows that constant relative risk aversion requires a decreasing absolute risk aversion, and that decreasing relative risk aversion requires an even more decreasing absolute risk aversion. Thus in all cases the rich gamble and the poor don't. (Note here that they are initially rich. Since they loose money on average they will become poor and stop gambling.)

iii) If $v(w) = \ln w$ then $v'(w) = 1/w$ and $v''(w) = -1/w^2$. Therefore $r_A = 1/w$, with $\partial r_A/\partial w < 0$, and $r_R = 1$. If $v(w) = \sqrt{w}$ then $v'(w) = 1/2\sqrt{w}$ and $v''(w) = -w^{3/2}/4$. Therefore $r_A = 1/(2w)$ and $r_R = 1/2$. We now know two pieces of information: the consumers' risk aversion to a given size gamble is declining with wealth. This would, *ceteris paribus* make them more likely to purchase the gamble for a constant μ (see above). But μ now is also declining with wealth. The final outcome therefore depends on what declines faster, and we can't make a definite statement. (As an aside note the following. Suppose we are talking stock market participation here. Then it might be reasonable to assume that the utility of participating in it is increasing in wealth, on average, and so we get higher participation by wealthier people. Now, if the stock market on average is a bad bet we get mean reversion in wealth, while if the stock market is on average more profitable than savings accounts etc we get the rich getting richer. If you now run a voting model where the mean voter wins, you get the desire to redistribute (i.e., tax the investing and profiting rich and give the cash to those who have a too high marginal utility of wealth to invest themselves.) Note also that progressive taxes reduce the returns of a given investment proportional to wealth, counteracting the above effect of more participation by wealthy individuals. ...

See how much fun you can have with these simple models and a willingness to extrapolate wildly?)

Question 7: This question forms part of a typical incomplete information contracting environment. Here we focus only on the consumer's behaviour.

a) Assume that the worker has a contractual obligation to provide an effort level of E. Once he has signed the contract, however, he knows that his actual effort is not observable and thus would try to shirk. Expected utility is maximized for

$$e^* = \text{argmax}\{\alpha\sqrt{w(E) - p} + (1-\alpha)\sqrt{w(E)} - e^2\}.$$

The first order condition for this problem is $-2e = 0$ **if** $e \neq E$. I.e., given the worker shirks he will go all the way (after all, the punishment does not depend on the severity of the crime in any way.) Thus we need to ensure that the worker will not shirk at all, which is the case if $\sqrt{w(E)} - E^2 \geq \alpha\sqrt{w(E) - p} + (1-\alpha)\sqrt{w(E)}$, or $E^2 \leq \alpha(\sqrt{w(E)} - \sqrt{w(E) - p})$. If the wage function satisfies this inequality for all E, it will elicit the correct effort levels in all cases.

b) Now we have a potentially variable punishment. Given some job with contractual obligation E, the worker now will maximize expected utility and set

$$e^* = \text{argmax}\{\alpha\sqrt{w(E) - p(E-e)} + (1-\alpha)\sqrt{w(E)} - e^2\}.$$

The FOC for this problem is $\alpha p'(\cdot)(2\sqrt{w(E) - p(E-e)})^{-1} - 2e = 0$. (There are also second order conditions which need to hold!) This implies that the worker will play off the cost of shirking against the gains from doing so. We need to make sure that this equation is only satisfied for $e^* = E$, in which case he "voluntarily" chooses the contracted level. This clearly requires a positive $p'()$. In particular, $\alpha p'(0)(2\sqrt{w(E)})^{-1} - 2E = 0$. Note: We could also vary the detection/supervision probability and make α depend on E. Then we get $e^* = \text{argmax}\{\alpha(E)\sqrt{w(E) - p} + (1-\alpha(E))\sqrt{w(E)} - e^2\}$. As in (a), if the worker deviates he will go all the way here. So the problem is similar to (a), only the wage schedule is now different since $\alpha(E)$ can also vary now. What this shows us is that we tend to want a punishment and a detection probability which both depend on the deviation from the correct level. (This is going to be a question about the technology available: some technologies may be able to detect flagrant shirking more readily than slight shirking.)

c) What this seems to indicate is that we would like to make punishments fit the crime. (So for example, if the punishment for a hold-up with a weapon is as severe as if somebody actually gets shot during it, then I might as well shoot people when I'm at it and I think that helps (and if it does not increase the effort the police put into finding me.)) Furthermore, if detection is a function of the actual effort level (the more you fudge the books the more likely will you be detected) then we need lower punishments, *ceteris paribus*, since the increasing risk will provide some disincentive to cheat anyways.

Question 8:

a) Let C_B denote the coverage purchased for bad losses, and C_M the coverage for minor losses. Zero profits imply that the premiums p_B and p_M for bad and minor losses, respectively, are $p_B = \pi/5$ and $p_M = 4\pi/5$. Hence the consumer's expected utility maximization problem becomes

$$\max_{C_B,C_M}\Big\{(1-\pi)u(W - p_M C_M - p_B C_B)+$$
$$\pi(\frac{1}{5}u(W - p_M C_M - p_B C_B + C_B - B)+$$

$$\left.\frac{4}{5}u(W - p_M C_M - p_B C_B + C_M - M))\right\}$$

The first order conditions for this problem are

$$\begin{aligned} -p_M(1-\pi)u'(n) - p_M\frac{\pi}{5}u'(b) + (1-p_M)\frac{4\pi}{5}u'(m) &= 0 \\ -p_B(1-\pi)u'(n) + (1-p_B)\frac{\pi}{5}u'(b) - p_B\frac{4\pi}{5}u'(m) &= 0 \end{aligned}$$

Using the fair premiums this simplifies to

$$\begin{aligned} -(1-\pi)u'(n) - \frac{\pi}{5}u'(b) + \left(1 - \frac{4\pi}{5}\right)u'(m) &= 0 \\ -(1-\pi)u'(n) + \left(1 - \frac{\pi}{5}\right)u'(b) - \frac{4\pi}{5}u'(m) &= 0 \end{aligned}$$

Hence

$$\left(1 - \frac{4\pi}{5}\right)u'(m) - \frac{\pi}{5}u'(b) = \left(1 - \frac{\pi}{5}\right)u'(b) - \frac{4\pi}{5}u'(m)$$

and thus $u'(b) = u'(m)$, which finally implies that $u'(n) = u'(b) = u'(m)$ and therefore that

$$0 = C_B - B = C_M - M.$$

As expected, the consumer buys full insurance for each accident type separately.

b) Now only one coverage can be purchased, denote it by C, and will be paid in case of either accident. Zero profits imply that the premium p is $p = \pi$. Hence the consumer's expected utility maximization problem becomes

$$\max_C \left\{(1-\pi)u(W - pC) + \pi(\frac{1}{5}u(W - pC + C - B) + \frac{4}{5}u(W - pC + C - M))\right\}$$

The first order condition for this problem is

$$-p(1-\pi)u'(n) + (1-p)\frac{\pi}{5}u'(b) + (1-p)\frac{4\pi}{5}u'(m) = 0$$

Using the fair premium this simplifies to

$$u'(n) = \frac{1}{5}u'(b) + \frac{4}{5}u'(m)$$

and hence either

$$u'(b) > u'(n) > u'(m) \quad \text{or} \quad u'(b) < u'(n) < u'(m).$$

Thus either $W - B + (1-\pi)C < W - \pi C < W - M + (1-\pi)C$ or $W - B + (1-\pi)C > W - \pi C > W - M + (1-\pi)C$, but this implies either $-B + 1C < 0 < -M + 1C$ or $-B + 1C > 0 > -M + 1C$. Since $B > M$ by definition we obtain that $B > C > M$, the consumer over insures against minor losses, and is under insured against big losses.

Question 9: From Figure 3.7 in the text, we can take the consumers' budget line to be the line from the risk free asset point (the origin in this case) to a tangency with the efficient portfolio frontier. Now this tangency occurs where the margin is equal to the average, so that $\sqrt{\sigma - 16}/\sigma = (2\sqrt{\sigma - 16})^{-1}$. That means that the market portfolio has $2(\sigma - 16) = \sigma$ or $\sigma = 32$. Therefore $\mu = 4$. The slope of the portfolio line thus is $4/32$. For an optimal solution the consumer's MRS must equal the slope of the portfolio line. For the two consumers given the MRS is $\sigma/32$ and $\sigma/96$. Thus the optima are $\sigma = 4,\ \mu = 1/2$ and $\sigma = 12,\ \mu = 3/2$. As expected, the consumer with the higher marginal utility for the mean will have a higher mean at the same prices (and given that both have the same disutility from variance.)

Question 10: The asset pricing formula implies that the expected return of the insurance equals the expected risk-free return less a covariance term. If insurance has a lower expected return than the risk-free asset, this covariance term must be positive. In the denominator we have the expected marginal utility, guaranteed to be positive. Thus the numerator must be positive. This means that $Cov(u'(w), R_i) > 0$. But since $u''(w) < 0$ this implies that the covariance between w and R_i is negative, that is, if wealth is low the return to the policy is high, if wealth is high, the return to the policy is low. That of course is precisely the feature of disability insurance which replaces income from work if and only if the consumer is unable to work.

Question 11:

1) False. The second order condition would indicate a minimum as demonstrated here: $max_C\{\pi u(w - L - pC + C) + (1-\pi)u(w - pC)\}$ has FOC $\pi(1-p)u'(w - L + (1-p)C) - p(1-\pi)u'(w - pC) = 0$. The second order condition for a maximum is $\pi(1-p)^2 u''(w - L + (1-p)C) + p^2(1-\pi)u''(w - pC) \leq 0$. Note that $1 \geq \pi, p \geq 0$, so that the SOC requires $u''(\cdot)$ to be negative for at least one of the terms. A risk-lover has, by definition, $u''(\cdot) > 0$.

2) Uncertain. We can draw 2 diagrams to demonstrate. In both we have two intersecting budget lines, one steeper, one flatter. The flatter one corresponds to the initial situation. They intersect at the consumer's endowment. Since the consumer is a borrower, the initial consumption point is below and to the right of the endowment on the initial budget. The indifference curve through this point is tangent to this budget. It may, however, cut the new budget (so that the IC tangent to the new budget represents a higher level of utility) or lie everywhere above it (in which case utility falls.)

3) True. Apply the following positive monotonic transformations to the first function: -2462, $\times 12$, collect terms in one logarithm, take exponential, take the 9000th root. What you get is the second function.

4) True. A risk averse consumer is defined as having $u(\int xg(x)dx) > \int u(x)g(x)dx$. Let the consumer have initial wealth w and suppose he could participate in a lottery which leads to a change in his initial wealth by x, distributed as $f(x)$. Suppose the payment for this lottery is p. If this payment is equal to the expected value of the lottery then the consumer will not have a change in expected wealth, but will face risk. Thus by definition he would not buy this lottery. If the payment is less, then the expected value of wealth from participating in the lottery exceeds the initial wealth. Depending on by how much, the consumer may purchase. A risk loving consumer, of course, would already buy at when the expected net gain is zero. (This argument could be made more precise, and you should try to put it into equations!)

5) False. The market rate of return is 15%. Gargleblaster stock has a rate of return of $(117-90)/90 = 30\%$. This violates zero arbitrage.

6) True. All consumers face the same budget line in mean-variance space. At an interior optimum (and assuming their MRS is defined) they all consume on this line where the tangency to their indifference curve occurs. This may be anywhere along the line, depending on tastes, but the slope is dictated by the market price for risk.

Question 12:

a) Since workers work as bus driver and at a desk job we require

$$2\sqrt{40000} = \alpha 2\sqrt{44100-11700} + (1-\alpha)2\sqrt{44100}$$

Therefore

$$\alpha = \frac{\sqrt{44100}-\sqrt{40000}}{\sqrt{44100}-\sqrt{32400}} = \frac{210-200}{210-180} = \frac{1}{3}.$$

b) Since workers work on oil rigs and at a desk job we require

$$2\sqrt{40000} = 0.5 \times 2\sqrt{122500 - \mathit{Loss}} + 0.5 \times 2\sqrt{122500}.$$

Thus $400 = \sqrt{122500-\mathit{Loss}} + 350$ and hence $50 = \sqrt{122500-\mathit{Loss}}$ or $\mathit{Loss} = 120000$.

c) At fair premiums the workers will fully insure. That is, they suffer their expected loss for certain. For a bus driver the expected loss is $11700/3 = 3900$. Thus the bus driver wage must satisfy $2\sqrt{40000} = 2\sqrt{w-3900}$ and hence it is \$43900. For the oil rig worker the expected loss is \$60000, and their wages will fall to \$100000 under workers compensation. Note the the condition that workers take all jobs together with a fixed desk job wage fixes the utility level in equilibrium for workers. However, the wage premium for risky jobs will not have to be paid: the wages of the risky occupations fall which benefits the firms in those industries by lowering their wage costs. (This is why industries are in favour of workers' compensation.)

d) The average probability of an accident now is $0.4 \times 0.5 + 0.6 \times 1/3 = 0.4$. If we were to use this as a fair premium (but see below!) this premium is too high for bus drivers, who will under insure, and too low for oil rig workers, who will over insure. Indeed, the bus drivers will choose to buy insurance C_b such that $6\sqrt{44100 - 0.4C_b} = 8\sqrt{32400 + .6C_b}$ (Take the first order condition for $max_{C_b}\{(1/3)\sqrt{32400 + 0.6C_b} + (2/3)\sqrt{44100 - 0.4C_b}\}$, bring the $\sqrt{\ }$ from the denominator into the numerators and loose the 1/30 on both sides.) Thus we require $9(44100 - 0.4C_b) = 16(32400 + 0.6C_b)$, or $C_b = 10(9 \times 44100 - 16 \times 32400)/(6 \times 16 + 4 \times 9) = -9204$. What does this mean? It means that the bus drivers would like to bet on themselves having an accident buying negative amounts of insurance! (The ultimate in under insurance!) Note that the governments expected profit from bus drivers is $-0.4 \times 9204/3 + 1.2 \times 9204/3 = 2454.40 > 0$.

The oil rig workers would need to solve $max_{C_o}\{\sqrt{2500 + 0.6C_o} + \sqrt{122500 - 0.4C_o}\}$, which leads to $3\sqrt{122500 - 0.4C_o} = 2\sqrt{2500 + 0.6C}$ and thus $C_o = 182083.33$. Note that the govt looses money on them, since $(0.5 \times 0.4 - 0.5 \times 0.6) \times 182083.33 = -18208.30$.

Overall then the govt makes losses of $5810.68N$, where N is the number of workers in risky occupations. At the old wages both groups are better off (and thus there would be an influx of desk workers and a reallocation towards oil rigs.) In order to break even the insurance rates would have to be changed, in particular raised. It also seems that the govt would ban the purchase of negative insurance amounts. In which case the bus drivers would find it optimal to buy no insurance, and then premiums would have to be 0.5 for the govt to break even. This would be deemed unjust by all involved, and so in practice the govt forces all workers to buy a fixed amount of insurance!

In principle we could compute equilibrium wages if we treat the insurance purchase as a function of the wage. So, for example we know from the above that $C_b(w) = 10(9 \times w - 16 \times (w - 11700))/(6 \times 16 + 4 \times 9)$. We then solve for $s\sqrt{40000} = (2/3)\sqrt{w + (1 - 0.4)C_b(w)} + (4/3)\sqrt{w - 11700 - 0.4C_b(w)}$. The details are left to the reader.

The important point here is that it is important to charge the correct premiums. If that is not done things will work out funny. That in turn leads to real life plans which do not allow a choice — workers have to insure, the amount is dictated (often capped, that is, the insured amount is a function of the wage up to a maximum.) You can see that such plans can be quite complicated and that it can be quite complicated to figure out who would want to do what, what the distributional implications are, etc.

Question 13: Let us translate the question into notation: We are to show that $\frac{u'(c_1)}{u'(c_2)} = k$ if $\frac{c_2}{c_1} = \lambda$ if the function $u(\cdot)$ satisfies $\frac{u''(w)w}{u'(w)} = a, \ \forall w$.

$$\frac{u'(c_1)}{u'(c_2)} = k \text{ and } \frac{c_2}{c_1} = \lambda \quad \Longrightarrow \quad u'(c_1) = k(\lambda)u'(\lambda c_1).$$

If the left and right hand side of that last expression are identical functions, then their derivatives must equal: $u''(c_1) = k(\lambda)\lambda u''(\lambda c_1)$, but we know that $k(\lambda) = u'(c_1)/u'(\lambda c_1)$, so that

$$u''(c_1) = \frac{u'(c_1)}{u'(\lambda c_1)}\lambda u''(\lambda c_1) \quad \Longrightarrow \quad \frac{u''(c_1)}{u'(c_1)} = \lambda\frac{u''(\lambda c_1)}{u'(\lambda c_1)}$$

Thus the MRS is constant for any consumption ratio λ if

$$\frac{u''(c_1)c_1}{u'(c_1)} = \frac{u''(\lambda c_1)\lambda c_1}{u'(\lambda c_1)} \quad \forall\lambda,$$

which is constant relative risk aversion.

7.4 Chapter 6

Question 1: A 3-player game in extensive form comprises a game tree, Γ, a payoff vector of length three for each terminal node, a partition of the set of non-terminal nodes into player sets S_0, S_1, S_2, S_3, a partition of the player sets S_1, S_2, S_3 into information sets. Further, a probability distribution for each node in S_0 over the set of immediate followers and for each S_i^J an index set I_i^J and a 1-1 mapping from I_i^j to the set of immediate followers of the nodes in S_i^j. Any carefully labelled game tree diagram will do. It does not even have to have nature (i.e., S_0 could be empty.)

Question 2: Perfect recall is when each player never forgets any of his own previous moves (so that for any two nodes within an information set one may not be a predecessor of the other and any two nodes may not have a common predecessor in another information set of that player such that the arc leading to the nodes differs) and never forgets information once known (so that any two nodes in a player's information set may not have predecessors in distinct previous information sets of this player.) Counter examples as in the text, or any game which violates these requirements.

Question 3: Yes, any finite game has a Nash equilibrium, possibly in mixed strategies. This follows from the Theorem we have in the text. The game therefore will also have a SPE (they are a subset of the Nash equilibria, but keep in mind that the condition of subgame perfection may have no 'bite', in which case we revert to Nash.)

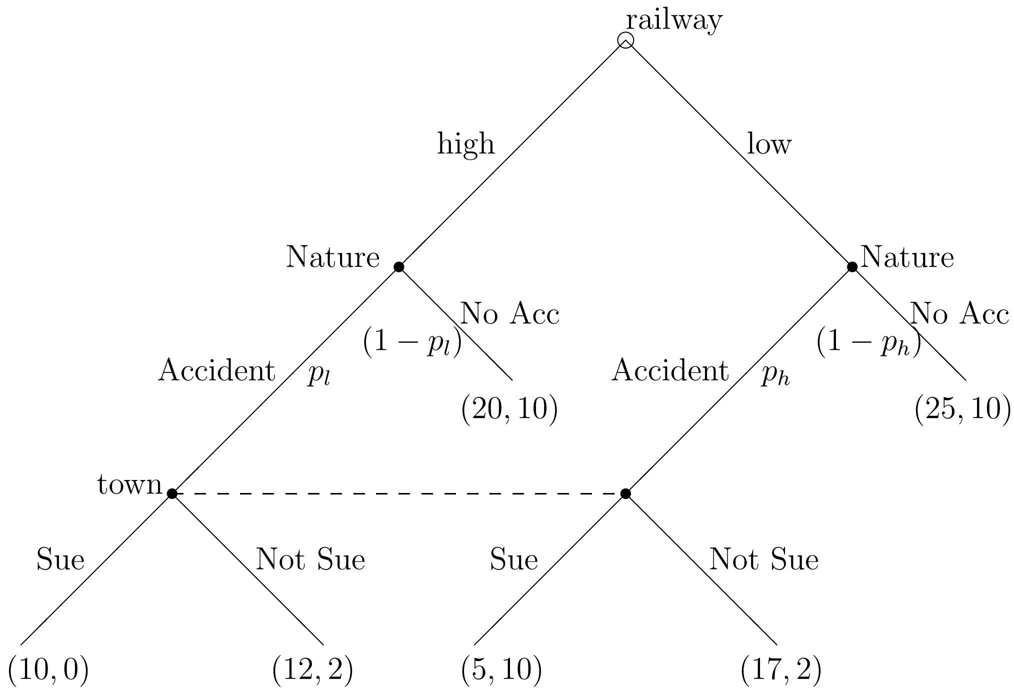
Question 4: Player 1 has no weakly dominated strategies since $u_1(D, L, Left) > u_1(U, L, Left)$ but $u_1(D, R, Left) < u_1(U, R, Left)$, while $u_1(C, L, Right) > u_1(U, L, Right)$. Player 3 does also not have a weakly dominated strategy. Depending on the opponents' moves he gets a higher payoff sometimes in the left and sometimes in the right matrix. Player 2 does have weakly dominated strategies: Both L and R are weakly dominated by C.

This does not leave us with a good prediction yet, aside from the fact that 2 can be argued to play C. However, if we now consider repeated elimination we can narrow down the answer to what is also the unique Nash equilibrium in pure strategies in this case, $(D, C, Right)$.

To find mixed strategy Nash we assign probabilities to the strategies for players, so let $\mu_1 = Pr(U), \mu_2 = Pr(C), \gamma_1 = Pr(L), \gamma_2 = Pr(R)$, and $\alpha = Pr(Left)$. We can then compute the payoffs for players for each of their pure strategies. So for example $u_1(U, \gamma, \alpha) = \alpha(\gamma_1 + 2\gamma_2 + (1 - \gamma_1 - \gamma_2) + (1 - \alpha)(2\gamma_1 + 4\gamma_2 + 2(1 - \gamma_1 - \gamma_2))$. We then can ask, when is player 1, say, actually willing to mix? Only if the payoff from the pure strategies in the support of the mixed strategy are equal, so that the player does not care.

Question 5: This one is made easier by the fact that strategy R is (strictly) dominated, so that it will never be used in any mixed strategy equilibrium (or indeed any equilibrium.) Hence this is really just a 2×2 matrix we need to consider. Let $\alpha = Pr(U)$ and $\beta = Pr(L)$, so that $Pr(C) = 1 - \alpha$ and $Pr(C) = 1-\beta$. Then for player 1 to mix we require $\beta+4(1-\beta) = 3\beta+2(1-\beta)$, hence $2 = 4\beta$, and hence $\beta = 0.5$. So if player 2 mixes with this probability then player 1 is indifferent between his two strategies. Now look at player 2: For 2 to be indifferent between the two strategies L and C we require $4\alpha + 2(1 - \alpha) = 2\alpha + 3(1 - \alpha)$. Hence $3\alpha = 1$ and thus $\alpha = 1/3$. Thus the mixed strategy Nash equilibrium is $((1/3, 2/3), (1/2, 1/2))$. Note that the game has no pure strategy Nash equilibria.

Question 6: This is a two player game (the court is not a strategic player and does not receive any payoffs.) The most natural extensive form for such a situation is probably as in the game tree on the next page.



Here it is important to note that $p_h > p_l$, reflecting the fact that if low care is taken the accident probability is higher. I have arbitrarily assigned payoffs which satisfy the description. High level of care costs the railway 5, accidents impose a cost of 8 on both parties, legal costs are 2 for each party.

Let us try and find the Nash equilibrium of this game. As an exercise let us first find the strategic form:

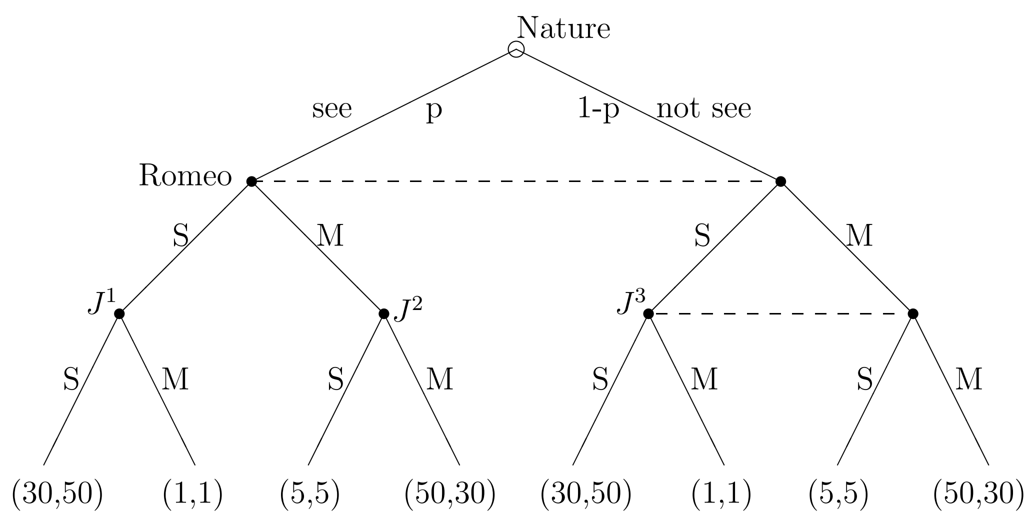
$R\backslash T$	*Sue*	*NotSue*
high	$(20 - 10p_l, 10 - 10p_l)$	$(20 - 8p_l, 10 - 8p_l)$
low	$(25 - 20p_h, 10)$	$(25 - 8p_h, 10 - 8p_h)$

Note that *low* strictly dominates *high* for the railway if $0.5 > (2p_h - p_l)$ while *high* strictly dominates *low* if $p_h - p_l > 5/8$. In those cases the

Nash equilibria are (low, Sue) and $(high, NotSue)$, respectively. Otherwise there will be a mixed strategy equilibrium. Let α be the probability with which the railway uses the high effort level. The town is indifferent iff its expected payoffs from the two strategies are the same, that is, if $10-10\alpha p_l = 10 - 8p_h + 8\alpha(p_h - p_l)$. This is the case if $\alpha = 4p_h/(4p_h + p_l)$. For lower α it prefers to Sue, for higher α it prefers to $NotSue$. Letting β denote the probability with which the town sues, the railway expects to receive $20 - 8p_l - 2\beta p_l$ from $high$ and $25 - 8p_h - 12\beta p_h$ from low. It is indifferent if $\beta = (5-8(p_h - p_l))/(12p_h - 2p_l)$. So the mixed strategy Nash equilibrium is

$$(\alpha,\beta) = \left(\frac{4p_h}{4p_h + p_l}, \frac{5-8(p_h-p_l)}{12p_h - 2p_l}\right) \qquad \text{if} \quad p_h - \frac{5}{8} > p_l > 2p_h - \frac{1}{2}.$$

Question 7: There are two ways to draw this game. We can have nature move first and then Romeo (who does not observe nature's move.) Or we can have Romeo move first, and then nature determines if the move is seen. The game tree for the first case is as drawn below.



A strategy vector in this game is $(s_R, (s_J^1, s_J^2, s_J^3))$. Subgames start at information sets J^1 and J^2, the only other subgame is the whole tree. In the subgame perfect equilibrium Juliet therefore is restricted to $(S, M, \cdot)$. Let α denote Romeo's probability of moving S, and β Juliet's (in J^3.) Romeo's (expected) payoff from S is $30p + (1-p)(30\beta + 1 - \beta)$ and his payoff from M is $50p + (1-p)(5\beta + 50(1-\beta))$. The β for which he is indifferent is $(49 - 29p)/(74(1-p))$. Note that this is increasing in p and that $\beta = 1$ if $p = 5/9$! Juliet has payoffs of $50\alpha + 5(1-\alpha)$ and $\alpha + 30(1-\alpha)$ from moving S

and M, respectively, in J^3. Hence she is indifferent if $\alpha = 25/74$. Of course, pure strategy equilibria may also exist, and we get the SPE equilibria to be

$$\left(\frac{25}{74}, \left(S, M, \frac{49-29p}{74(1-p)}\right)\right),\ (S,(S,M,S))\,,\ (M,(S,M,M))\ \text{ if } p < \frac{5}{9}.$$

Note that $(S,(S,M,S))$ requires that $30 > 50p + 5(1-p)$, or $p < 5/9$ also. $(M,(S,M,M))$ requires that $50 > 30p+(1-p)$, or $p < 49/29$, which is always true. What if $p > 5/9$? In that case the equilibrium in which the outcome is coordination on S (preferred by Juliet) does not exist, and neither does the mixed strategy equilibrium. Hence the unique equilibrium if $p \geq 5/9$ is $(M,(S,M,M))$. Romeo can effectively insist on his preferred outcome.

Question 8: Each firm will

$$\max_{q_i} \left\{ (\sum_{j\neq i} q_j + q_i - 10)^2 q_i - 0q_i \right\}.$$

The FOC for this problem is

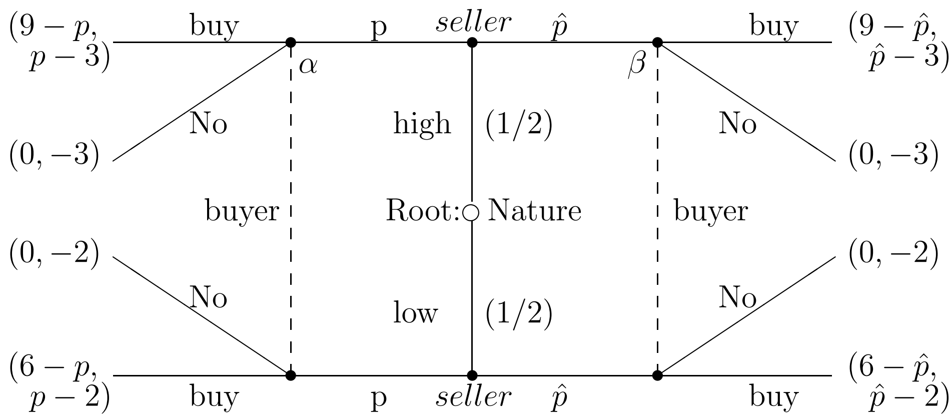
$$2(\sum_{j\neq i} q_j + q_i - 10)q_i + (\sum_{j\neq i} q_j + q_i - 10)^2 = 0$$

Hence if $\sum_{j\neq i} q_j + q_i - 10 \neq 0$ we require

$$2q_i + (\sum_{j\neq i} q_j + q_i - 10) = 0$$

and get the reaction function $q_i = (10 - \sum_{j\neq i} q_j)/3$. With identical firms we then know that in equilibrium $q_j = q_i$, so that $\sum_{j\neq i} q_j = (n-1)q_i$. Hence $3q_i = 10 - (n-1)q_i$ and we get that $q_i* = 10/(n+2)\ \forall i$. Total market output then is $10n/(n+2)$. Note that total market output approaches 10 from below as n gets large. Market price for a given n is $400/(n+2)^2$, which approaches zero as n gets large. (Note that the marginal cost is zero and hence the perfectly competitive price is zero!)

Question 9: In the first instance the sellers can only vary price. To clarify ideas, let us focus on two prices only (as would be needed for a separating equilibrium.) The game then is as depicted below. We are to show that no separating equilibrium exists. If it did, it would have to be the two prices as indicated, where one firm charges one price (presumably the high quality firm charging the higher price) the other another. But given that, the consumer knows (in equilibrium) which firm produced the product. It is easy to see that the low quality firm would deviate to the higher price (being then mistaken



for the high quality firm so that the consumer buys) since costs are unaffected by such a move, but a higher price is received.

At this point the remainder is non-trivial and left for summer study! The key is that the consumer now has an information set for each price-warranty pair, and that there are two nodes in it, one for each type of firm.

Question 10: What was not stated in the question was the fact that each consumer buys either one or no units. Each buyer purchases a unit of the good if and only if the price is below the valuation of the buyer. Hence total market demand is given by the number of buyers with a valuation above p, or $1 - F(p)$. $F(v)$ is the cumulative distribution for the uniform distribution on $[0, 2]$. Since the pdf for the uniform distribution on $[0, 2]$ is 0.5, we have $F(v) = \int_0^v 0.5dt = 0.5v$. Hence market demand is $1 - 0.5p$ and inverse market demand is $2(1 - Q)$.

A Cournot equilibrium is nothing but a Nash equilibrium in the game in which firms simultaneously choose output levels. Hence firm 1 solves

$$max_{q_1} \{2(1 - q_1 - q_2)q_1 - q_1/10\}$$

which leads to FOC $2(1 - q_1 - q_2) - 2q_1 - 1/10 = 0$ and the reaction function $q_1(q_2) = 19/40 - q_2/2$.

Firm 2 solves

$$max_{q_2} \left\{2(1 - q_1 - q_2)q_2 - q_2^2\right\}$$

which leads to FOC $2(1 - q_1 - q_2) - 2q_2 - 2q_2 = 0$ and the reaction function $q_2(q_1) = 1/3 - q_1/3$.

The Nash equilibrium then is $(q_1, q_2) = (37/100, 21/100)$. Market price is $42/50$. Profits for the two firms are $74 \times 37/10000$ for firm 1 and $(82 \times 21 - 21^2)/10000$ for firm 2, so that joint profit is $(74 \times 37 + 82 \times 21 - 21^2)/10000$.

In the Stackelberg leader case we consider the SPE of the game in which firm 1 chooses output first and firm 2, after observing firm 1's output choice, picks its output level. Firm 1, the Stackelberg leader, therefore takes firm 2's reaction function as given. Thus firm 1 solves

$$max_{q_1} \left\{ 2\left(1 - q_1 - \left(\frac{1}{3} - \frac{q_1}{3}\right)\right) q_1 - q_1/10 \right\}$$

The FOC for this is $4(1-2q_1)/3 - 1/10 = 0$ and hence $q_1 = 37/80$. Thus $q_2 = 43/240$. Market price is $86/240$. Profits for the two firms are $(62 \times 37)/(240 \times 80)$ and $43 \times 43/240^2$. Joint profit thus is $(186 \times 37 + 43 \times 43)/240^2$.

Joint profit maximization would require that the firms solve

$$max_{q_1,q_2} \left\{ 2(1 - q_1 - q_2)(q_1 + q_2) - q_1/10 - q_2^2 \right\}.$$

This has FOCs

$$\begin{aligned} 2(1 - q_1 - q_2) - 2(q_1 + q_2) - 1/10 &= 0 \\ 2(1 - q_1 - q_2) - 2(q_1 + q_2) - 2q_2 &= 0 \end{aligned}$$

so that we know that $2q_2 = 1/10$ or $q_2 = 1/20$. Hence $2(1 - q_1 - 1/20) - 2(q_1 + 1/20) - 1/10 = 0$ and $2 - 4q_1 - 3/10 = 0$ and $q_1 = 7/40$. Market price then is $2(31/40)$. Joint profits are $2(31/40)(9/40) - 8/400$. This cannot be attained as a Nash equilibrium because neither output level is on the firm's reaction function, and only output levels on the reaction function are, by design, a best response.