Introduction to Bond Valuation

(Text reference: Chapter 5 (Sections 5.1-5.3, Appendix))

Topics

- types of bonds
- valuation of bonds
- yield to maturity
- term structure of interest rates
- more about forward rates

Types of Bonds

- a bond is a form of debt (i.e. a contractual liability; basically just a certificate showing that a borrower promises to repay interest and principal on specified dates
- issued by both governments and corporations
- example: the Government of Canada issued a bond with a face value of $1,000 in June 2002 which matures in June 2022. The stated annual interest rate is 8%:
  - the face value is $1,000 (a.k.a. the principal or par value)
  - the annual coupon is $80
  - the coupon rate is 8%
  - the time to maturity is 20 years
  - the maturity date is June 1, 2022
there are many varieties of bonds:

- a pure discount bond (a.k.a. zero coupon bond) pays the bond’s face value at maturity (and nothing else)
- a consol pays a stated coupon at periodic intervals and has no maturity date
- a level coupon bond pays the bond’s face value at maturity and a stated coupon at periodic intervals prior to maturity
- a callable bond gives the issuer the right to buy the bond back before maturity for specified price(s) on specified date(s)
- a convertible bond allows the owner to exchange it for a specified number of shares of stock

Valuation of Bonds

- at the time of issue a level coupon bond is usually sold for a price which is close to its par value
- after issue a bond is traded on the market at a price which reflects the current level of interest rates and the degree of risk associated with the bond
- typically we are interested in calculating either the market price that a bond should sell for, given that the investor wants to obtain a particular market yield; or the effective yield (a.k.a. the yield to maturity), given the price at which the bond is trading
- the value of a financial security is the PV of expected future cash flows ⇒ to value bonds we need to estimate future cash flows (size and timing) and discount at an appropriate rate
Cont’d

- notation: coupon payment \( C \), face value \( F \), discount rate \( r \), time to maturity \( T \)

- value of a consol:

\[
PV \text{ of consol} = \frac{C}{r}
\]

- value of a pure discount bond:

\[
PV \text{ of pure discount bond} = \frac{F}{(1 + r)^T}
\]

- value of a level coupon bond:

\[
PV \text{ of level coupon bond} = C \times \left[ \frac{1 - (1 + r)^{-T}}{r} \right] + \frac{F}{(1 + r)^T}
\]

\[
= C \times A_r^T + \frac{F}{(1 + r)^T}
\]

Cont’d

- example: consider a $1,000 par value bond with 17 years remaining until maturity and a coupon rate of 6% 
- what is the price of this bond if market interest rates are 8%?

- suppose an investor buys this bond at this price and holds it for one year. If interest rates remain at 8%, what rate of return has the investor earned?
Cont’d

suppose instead interest rates fall to 6%:

suppose instead interest rates rise to 10%:

when interest rates rise, market prices of bonds fall (and vice versa) (the longer the time until maturity, the more sensitive the bond price is to changes in interest rates)

if price < par value, a bond is said to sell at a discount

if price > par value, a bond is said to sell at a premium

if price = par value, a bond is said to sell at par

in practice most bonds pay interest semi-annually, so we have to find the appropriate semi-annual rate and adjust coupon payments. For example, consider the bond above (top of slide [6]) but with semi-annual payments, and assuming that the 8% annual rate is a stated rate (not the effective annual rate):
Yield to Maturity

- the *yield to maturity* of a bond is the discount rate which equates the price of a bond with the PV of its expected future cash flows.

- example: what is the yield to maturity of a 5% coupon 9 year $1,000 par value bond if the price is $813 (annual coupons)? We need to solve the following equation for $r$:

$$813 = \sum_{t=1}^{9} \frac{50}{(1+r)^t} + \frac{1000}{(1+r)^9}$$

Cont’d

- suppose instead coupons are semi-annual:

$$813 = \sum_{t=1}^{18} \frac{25}{(1+r)^t} + \frac{1000}{(1+r)^{18}}$$
Term Structure of Interest Rates

- what determines interest rates?
  - investor preferences (e.g. willingness to save affects the supply of capital)
  - productive opportunities (e.g. firms’ desire to invest affects the demand for capital)
  - inflation (let \( i \) denote the expected rate of inflation)
- inflation erodes the purchasing power of money ⇒ distinguish between nominal and real interest rates:
  
  \[
  1 + r_{\text{nominal}} = (1 + r_{\text{real}}) \times (1 + i)
  \]

  ⇒ \( r_{\text{nominal}} = r_{\text{real}} + i + i \times r_{\text{real}} \approx r_{\text{real}} + i \)

- nominal rates will change whenever expected inflation does

Cont’d

- until now we have assumed that nominal interest rates as of today are the same for all future periods; this allowed us to use the general PV formula:

  \[
  PV = \sum_{t=1}^{T} \frac{C_t}{(1 + r)^t}
  \]

- but on May 11, 2005 we saw the following rates in the market:

<table>
<thead>
<tr>
<th>1-mo.</th>
<th>3-mo.</th>
<th>6-mo.</th>
<th>1-yr.</th>
<th>2-yr.</th>
<th>5-yr.</th>
<th>7-yr.</th>
<th>10-yr.</th>
<th>30-yr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.41</td>
<td>2.46</td>
<td>2.58</td>
<td>2.82</td>
<td>3.06</td>
<td>3.50</td>
<td>3.93</td>
<td>4.09</td>
<td>4.52</td>
</tr>
</tbody>
</table>

  (Source: National Post)

- the rates above are zero coupon rates, or spot rates
- the term structure of interest rates or yield curve is the relationship between spot rates of different maturities
Cont’d

- denote the \( t \)-period spot rate by \( r_t \). Then our general PV formula becomes:

\[
PV = \sum_{t=1}^{T} \frac{C_t}{(1 + r_t)^t}
\]

- our earlier formula is a special case of this with a “flat term structure” (i.e. \( r_1 = r_2 = r_3 = \cdots = r \))

- rather than discounting each period's cash flow at a different rate, we could find a single rate which gives the same PV. This is just the yield to maturity from above (more generally, it is known as internal rate of return (IRR))

Cont’d

- some problems with yield to maturity:
  1. The same rate is used to discount all payments, but what if \( r_1 \neq r_2 \neq r_3 \cdots \)? For example:

    Bond A: 3 years, annual coupon of 5%
    Bond B: 3 years, annual coupon of 15%
    Spot rates: \( r_1 = .04, r_2 = .048, r_3 = .054 \)

    Calculate the yield to maturity and PV for each bond:
Cont’d

2. Yields to maturity do not add up—even if you know the yield to maturity for A and the yield to maturity for B, you do not know the yield to maturity for A plus B.

3. Yield to maturity can give a misleading impression of return since it implicitly assumes that all intermediate payments are reinvested and earn the same rate of return.

Cont’d

how can we measure the term structure?

1. Calculate yields to maturity on zero coupon bonds.
2. Calculate spot rates from coupon bonds with the same maturity and different coupon rates. Consider our earlier example (slide 14) with Bonds A and B:
Cont’d

explaining the term structure: Assume \( r_1 = 5\% \), \( r_2 = 4\% \).
Suppose I want to invest $100 for two years. I can:

(a) buy a two year bond:

\[
100 \times 1.04^2 = 108.16
\]

or \( 100 \times 1.05 \times (1 + f_2) = 108.16 \Rightarrow f_2 = 0.0301 \)

where \( f_2 \) is the forward rate implied by the two year spot rate

(b) buy a series of one year bonds:

\[
100 \times 1.05 \times (1 + r_{1,2})
\]

where \( r_{1,2} \) is the one year spot rate next year

Cont’d

(a) provides a certain payoff of $108.16, whereas (b) provides an expected payoff of $100 \times 1.05 \times [1 + E(r_{1,2})]

suppose instead that I want to invest $100 for one year. I can:

(c) buy a one year bond: $100 \times 1.05 = $105

(d) buy a two year bond and sell it after one year—what is the price of a one year bond a year from now?

\[
P_{\text{V}} = \frac{100(1 + r_2)^2}{1 + r_{1,2}} = \frac{100(1 + r_1)(1 + f_2)}{1 + r_{1,2}}
\]

(c) provides a certain payoff of $105, while (d) provides an expected payoff of

\[
P_{\text{V}} = \frac{100(1 + r_1)(1 + f_2)}{1 + E(r_{1,2})}
\]
if all investors try to maximize expected return, then

\[ f_2 = E(r_{1,2}) \]

Why?

d this is called the **expectations hypothesis**: forward rates are equal to expected future spot rates. The term structure slopes up if expected future spot rates are higher than current spot rates, and vice versa.

Cont’d

note that the expectations hypothesis ignores risk. One alternative is the **liquidity preference hypothesis**: the shape of the term structure depends on the investment horizons of investors.

suppose investors have a short term horizon, e.g. one year. This implies that short term bonds are less risky than long term bonds (compare (c) and (d) above) ⇒ to induce investors to hold long term bonds, they must receive a risk premium.

suppose instead investors have a long term horizon. In this case, long term bonds are less risky than short term bonds (compare (a) and (b) above) ⇒ to induce investors to hold short term bonds, they must receive a risk premium.
More About Forward Rates

- start with a foreign currency example. On May 11, 2005, the National Post reported that the spot exchange rate between $CAD and $US was 1.2518, and the one year forward exchange rate was 1.2429.

- to determine the forward rate, we also need one year T-bill rates in each country; the National Post reported the Cdn rate was 2.82%; the U.S. rate was not given, but assume 3.56%

- a Canadian investor with $100 to invest for one year can make a risk free investment in Canada:

  - alternatively:

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- equilibrium requires that these two risk free strategies offer the same rates of return:

\[ 1 + r_C = \frac{1}{E_0} \times (1 + r_F) \times F_1 \]

where \( r_C \) is the one year Cdn T-bill rate, \( E_0 \) is the spot exchange rate, \( r_F \) is the one year US T-bill rate, and \( F_1 \) is the one year forward rate

- the equation above is called spot-forward parity

- given any three of the four variables \( (r_C, r_F, E_0, F_1) \), we can always find the fourth using spot-forward parity
Cont’d

- now consider forward interest rates: these aren’t quoted in the financial press, but they are established in financial markets from prices of zero coupon bonds (i.e. the term structure)

- a simple example: suppose a one year zero coupon bond ($1,000 par value) is selling for $952.38 and a two year zero coupon bond is selling for $902.73

- consider the cash flows from buying a one year zero coupon bond today and simultaneously selling $952.38/902.73 = 1.055 two year zero coupon bonds:

Cont’d

- we can see from this example that there is no cash flow today, but the investor has effectively locked in a rate to borrow money after one year for repayment after two years

- recall from earlier that 
  \[(1 + r_2)^2 = (1 + r_1)(1 + f_2),\]
  implying 
  \[f_2 = (1 + r_2)^2 / (1 + r_1)\]

- in this case: