Futures and Forward Markets

(Text reference: Chapters 19, 21.4)

- background
- hedging and speculation
- optimal hedge ratio
- forward and futures prices
- futures prices and expected spot prices
- stock index futures
- interest rate futures
- futures contracts in the binomial model
- the Value Line stock index futures contract—a history lesson

Background

- a *forward contract* is an agreement to buy/sell an asset at a particular future time for a certain price
- forward contracts are customized and not usually traded on an exchange
- note that swaps are basically just portfolios of forward contracts
- the *long (short)* position agrees to buy (sell) the asset on the specified date for the *delivery price*
- when the contract is entered into, the delivery price is chosen so that the value of the contract is zero to each party
- the *forward price* is the delivery price which makes the contract value zero (⇒ forward price is equal to delivery price at inception)
- let $S_T$ be the spot price of the asset at $T$ and $F_0$ be the delivery price. Then the payoff to the long position is $S_T - F_0$ and the payoff to the short position is $F_0 - S_T$. 
- a *futures contract* is similar except that it is standardized (delivery location, contract size, etc.) and traded on an organized exchange.

- the exact delivery date is usually not specified in a futures contract; rather it is some time interval within the delivery month.

- actual delivery rarely occurs, instead parties close out positions by taking offsetting transactions prior to maturity.

- futures trading uses margin accounts to minimize default risk. Initial margin is required when a contract is entered. Each day contracts are marked-to-market, with gains/losses added to margin accounts. If the value of the margin account falls below the maintenance margin, the contract holder receives a margin call. If additional cash is not added to the account, then the contract will be closed out.

- the futures price will converge to the spot price of the underlying asset when the contract matures.
Hedging and speculation

- a speculator uses futures contracts to profit from price movements, whereas a hedger tries to protect against price movements

- a firm which will be selling an asset in the future can hedge by taking a short futures position (a short hedge). If the asset price falls, the firm loses on the asset sale but gains on the futures contract; if the asset price rises, the firm gains on the sale but loses on the futures contract.

- similarly, a firm which will purchase an asset can hedge by taking a long hedge

- note that futures hedging does not necessarily improve the overall outcome: you can expect to lose on the futures contract about half the time

reasons why futures hedging may not work perfectly:

- the asset being hedged may not be exactly the same as the asset underlying the futures contract
- the hedger may be uncertain as to exactly when the asset will be bought/sold
- the hedger may have to close out the futures contract before the expiration date

basis is defined as the spot price of the asset being hedged less the futures price of the contract used

- if the asset being hedged is the same as that underlying the futures contract, basis is zero at expiry (can be positive or negative before)

- assume a hedge is started at time $t_1$ and closed at time $t_2$, and let $S$ be the spot price of the asset, $F$ be the futures price, and $b$ be the basis

- for a short hedge, the effective price is

$$S_{t_2} + F_{t_1} - F_{t_2} = F_{t_1} + b_{t_2}$$
– for a long hedge, the effective cost is
\[ S_{t_2} + F_{t_1} - F_{t_2} = F_{t_1} + b_{t_2} \]

– \( F_{t_1} \) is known at \( t_1 \), but \( b_{t_2} \) is not \( \Rightarrow \) hedging risk is uncertainty associated with \( b_{t_2} \) (called \textit{basis risk})

– for investment assets (currencies, stock indexes), basis risk is normally quite low, but it can be significant for commodities

– if the asset being hedged is not the same as that underlying the futures contract, there are two components to basis risk
  * let \( S^* \) be the price of the asset underlying the futures contract
  * by hedging, the price paid/rec’d for the asset is
    \[ S_{t_2} + F_{t_1} - F_{t_2} = F_{t_1} + (S^*_{t_2} - F_{t_2}) + (S_{t_2} - S^*_{t_2}) \]

– aspects of contract choice:
  * asset underlying futures contract
  * delivery month (rule of thumb: close to but after expiry of hedge)

– example: On March 1 a firm learns it will receive 50 million yen on July 31. Yen futures contracts have delivery months of March, June, September, and December. One contract is for delivery of 12.5 million yen. The firm shorts 4 September futures contracts on March 1 and closes out its position on July 31.

  – suppose that the futures price on March 1 is 0.78 cents per yen and on July 31 it is 0.725 cents per yen. The spot exchange rate on July 31 is 0.72 cents per yen.

    \[ \Rightarrow \text{ basis } = 0.72 - 0.725 = -0.005 \]

    \[ \Rightarrow \text{ effective price obtained is final spot price plus gain on futures: } 0.72 + (0.78 - 0.725) = 0.775. \text{ This equals the initial futures price plus the final basis: } 0.78 - 0.005 = 0.775. \]

  – the firm receives 50 million yen \( \times 0.00775 = $387,500 \)
Optimal hedge ratio

- the *hedge ratio* is

\[
\frac{\text{size of position in futures market}}{\text{size of exposure}}
\]

- up to here we have simply assumed that this should be 1, but if the objective is to minimize risk, a different hedge ratio may be preferred

- let the hedge ratio be \( h \) and

\[
\Delta S = \text{change in spot price over hedge period} \\
\Delta F = \text{change in futures price over hedge period} \\
\sigma_S = \text{standard deviation of } \Delta S \\
\sigma_F = \text{standard deviation of } \Delta F \\
\rho = \text{correlation between } \Delta S \text{ and } \Delta F
\]

for a short hedge, the change in the value of the position is \( \Delta S - h\Delta F \); for a long hedge it is \( h\Delta F - \Delta S \)

- in either case, the variance of the change in the value of the position is

\[
V = \sigma_S^2 + h^2 \sigma_F^2 - 2h\rho \sigma_S \sigma_F
\]

\[
\Rightarrow \frac{\partial V}{\partial h} = 2h\sigma_F^2 - 2\rho \sigma_S \sigma_F
\]

- setting the derivative to zero gives the optimal value of \( h \) as

\[
h^* = \frac{\rho \sigma_S \sigma_F}{\sigma_F^2}
\]
**Forward and futures prices**

- Start with forward prices, assuming that the underlying asset provides a known income (e.g. coupon bond, known dividend, FX)

- Notation:

  \[ T = \text{maturity date of forward contract} \]
  \[ S_t = \text{spot price of asset at time } t \]
  \[ F_t = \text{forward price of asset at time } t \]
  \[ I_t = \text{PV}_t \text{ of income paid by asset from } t \text{ to } T \]
  
  (discounted at \( r_f \))

**Spot-futures parity:**

<table>
<thead>
<tr>
<th>A. Buy underlying asset</th>
<th>Initial Cash Flows</th>
<th>Cash Flows at ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Borrow ( I_0 ), repay using income rec’d from asset</td>
<td>(-S_0)</td>
<td>(S_T)</td>
</tr>
<tr>
<td>(I_0 - S_0)</td>
<td>(I_0)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. Long forward contract</th>
<th>(0)</th>
<th>(S_T - F_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lend ( \text{PV}_0(F_0) ) at ( r_f )</td>
<td>(-F_0/(1 + r_f)^T)</td>
<td>(F_0)</td>
</tr>
<tr>
<td>(-F_0/(1 + r_f)^T)</td>
<td>(-F_0/(1 + r_f)^T)</td>
<td>(S_T)</td>
</tr>
</tbody>
</table>

\[ \Rightarrow I_0 - S_0 = \frac{-F_0}{(1 + r_f)^T} \]  

\[ \text{or } F_0 = (S_0 - I_0)(1 + r_f)^T \]
• note that if the income is in the form of a dividend \( D \) and it is paid out at \( T \), we have
\[
F_0 = S_0(1 + r_f)^T - D
\]
as in the text (equation 19.1)

• other cases:
  - continuous dividend yield: \( F_0 = S_0e^{(r_f - q)T} \) where \( q \) is the dividend yield.
    In the FX case, \( q \) is the foreign risk free interest rate (continuously compounded).
    \[
    \Rightarrow F_0 = S_0 \frac{e^{r_f T}}{e^{r_f T}} = S_0 \frac{e^{r_f^D T}}{e^{r_f T}} = S_0 \frac{(1 + r_f^D)^T}{(1 + r_f)^T} \text{ with effective annual rates}
    \]

• cost-of-carry:
  * suppose it costs you (in PV terms) \( C_0 \) to store the asset. This is like a negative dividend \( \Rightarrow F_0 = (S_0 + C_0)(1 + r_f)^T \).

• convenience yield:
  * commodities such as copper and oil are inventoried for use in production so there is a benefit to holding the asset (called a convenience value)
  * let \( Y_0 \) be the PV of the convenience value from holding the asset
    \[
    \Rightarrow F_0 = [S_0 + C_0 - Y_0](1 + r_f)^T \text{ or } F_0 = [S_0 - Y_0^N](1 + r_f)^T
    \]
    where \( Y_0^N \) is the net convenience value \( Y_0 - C_0 \)
  * in continuous yield form \( q \) is the net convenience value at a continuous rate
• the initial value of the forward contract is zero, what about at $t < T$?

$$\text{PV}_t[S_T - F_0] = S_t - I_t - \frac{F_0}{(1 + r_f)^{T-t}} = \frac{F_t - F_0}{(1 + r_f)^{T-t}}$$

This gives the amount paid/received for closing out the contract at $t$. It depends on the new forward price/prevailing spot price and the level of $r_f$, and can be positive or negative.

• forward vs. futures prices:

  long forward payoff $= S_T - F_0$
  long futures payoff $= 0 + (F_1 - F_0) + (F_2 - F_1) + \cdots + (F_T - F_{T-1})$
  $= F_T - F_0$
  $= S_T - F_0$ by the convergence property

⇒ total payoff is the same, but there are differences in the timing of cash flows due to marking-to-market

• it can be shown that if interest rates are not random, forward and futures prices are equal

• futures prices will differ from forward prices when marking-to-market gives a systematic advantage to one party (long or short), e.g. if the futures price rises when interest rates are high, the long will tend to benefit and the futures price will exceed the forward price

• empirically the differences are quite small (except for some commodities)
Futures prices and expected spot prices

- expectations hypothesis: $F_0 = E(S_T)$. This assumes risk-neutrality and implies that the expected profit is zero for both long and short parties since neither requires a risk premium.

- normal backwardation: hedgers will accept a penalty in order to reduce risk, speculators require a reward for taking it on. Most hedgers are short hedgers so $E(S_T) > F_0$.

- contango: same idea, but assumes most hedgers are long hedgers so $E(S_T) < F_0$ to get speculators to take short positions

- more generally, let $k$ denote the required rate of return on an asset. In equilibrium

$$S_0 - I_0 = \frac{E(S_T)}{(1 + k)^T}$$

and this must be equal to

$$\text{PV}(F_0) = \frac{F_0}{(1 + r_f)^T}$$

$$\Rightarrow F_0 = E(S_T) \left( \frac{1 + r_f}{1 + k} \right)^T.$$ 

- thus for positive $\beta$ assets ($k > r_f$), $F_0 < E(S_T)$
Stock index futures

- cash settlement, for simplicity assume constant dividend yield so
  \[ F_0 = S_0 e^{(r_f-q)T} \]
  where \( S_t \) is the value of the index at time \( t \)

- index arbitrage:
  - if \( F_0 < S_0 e^{(r_f-q)T} \), short stocks in index and go long the index
    futures contract
  - if \( F_0 > S_0 e^{(r_f-q)T} \), short the index futures contract and go long the
    stocks
  - to minimize transactions costs, you may have to trade in a
    representative basket of securities (not pure arbitrage)
  - usually done via program trading

hedging:

- suppose you hold a portfolio with current value \( V_0 \). If the CAPM is
  correct, then
  \[ E(r_p) = r_f + \beta_P (E(r_M) - r_f), \]
  implying
  \[ E(V_1) = V_0 (1 + r_f (1 - \beta) + \beta E(r_M)). \]

- suppose you short \( N \) futures contracts on the index, so your overall
  expected payoff is
  \[ V_0 (1 + r_f (1 - \beta) + \beta E(r_M)) - NF E(r_M), \]
  where \( F \) is the value of the futures contract (usually 250 \( \times \) futures price)

- if you pick \( N = \beta V_0 / F \), the overall expected payoff becomes
  \[ V_0 (1 + r_f (1 - \beta) + \beta E(r_M)) - \frac{\beta V_0}{F} E(r_M) = V_0 (1 + r_f (1 - \beta)), \]
  so your portfolio should grow at approximately \( r_f (1 - \beta) \) with no
  exposure to market risk
- if this is what you want, why not sell the portfolio and purchase T-bills?
  1. The hedging strategy offers short term protection against market movements at low transactions costs.
  2. You may not want to sell your portfolio if you think that it contains undervalued stocks.

- see handout for example
- more generally, if we let $N = \hat{\beta}V_0/F$, the overall expected payoff becomes

$$V_0(1 + r_f (1 - \beta) + \beta E(r_M)) - \frac{\hat{\beta}V_0}{F} FE(r_M)$$

$$= V_0(1 + r_f (1 - \beta) + (\beta - \hat{\beta})E(r_M))$$

- futures contracts also allow you to take positions on the performance of one index relative to another (e.g. long S&P Midcap, short NYSE)

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**Interest rate futures**

- there are a variety of interest rate futures contracts (Canada bond, U.S. Treasury bonds, Eurodollar)
- most of these are really contracts on the prices of bonds (Eurodollar contracts are an exception—see text pp. 755-757 for a description)
- can be used for a variety of hedging/speculative purposes (e.g. temporarily immunizing a bond portfolio, increasing duration to take advantage of a forecasted drop in yields, etc.)
- general rule for immunization: the hedge ratio should be

$$H = \frac{\text{PVBP of portfolio}}{\text{PVBP of futures contract}}$$

(see text p. 831 for an example)

- note that in practice there are complicated procedures for determining which bonds may to be delivered to settle the futures contract
Futures contracts in the binomial model

- Recall our two-period example of valuing a call option on a stock with $S_0 = 100$, $r_f = .0488$, $u = \sqrt{1.3}$, $d = 1/u$, and $X = 98$:

\[
\begin{align*}
S_0 &= 100 \\
C_0 &= 13.22 \\
h &= .7348 \\
B &= -60.26
\end{align*}
\]

\[
\begin{align*}
S_u &= 114.02 \\
C_u &= 20.58 \\
h &= 1 \\
B &= -93.44
\end{align*}
\]

\[
\begin{align*}
S_d &= 87.71 \\
C_d &= 1.24 \\
h &= .0867 \\
B &= -6.36
\end{align*}
\]

\[
\begin{align*}
S_{uu} &= 130 \\
C_{uu} &= 32
\end{align*}
\]

\[
\begin{align*}
S_{ud} &= 100 \\
C_{ud} &= 2
\end{align*}
\]

\[
\begin{align*}
S_{dd} &= 76.92 \\
C_{dd} &= 0
\end{align*}
\]

- Suppose instead we want to value a futures contract on the stock. By the convergence property, $F_{uu} = 130$, $F_{ud} = 100$, and $F_{dd} = 76.92$. What about $F_u$? We can do a similar replication argument:

\[
\begin{align*}
130h + 1.0488B &= 130 - F_u \\
100h + 1.0488B &= 100 - F_u \\
114.02h + B &= 0
\end{align*}
\]

- The right hand side of the first two equations above is the cash flow to the long futures position at the end of the year. The third equation is derived from the restriction that it costs nothing to enter a futures contract. The solution is $h = 1$, $B = -114.02$, $F_u = 119.58$. 


• similarly, for $F_d$:

$$
100h + 1.0488B = 100 - F_d \\
76.92h + 1.0488B = 76.92 - F_d \\
87.71h + B = 0.
$$

The solution is $h = 1, B = -87.71, F_d = 91.99$.

• finally, for $F_0$:

$$
114.02h + 1.0488B = 119.58 - F_0 \\
87.71h + 1.0488B = 91.99 - F_0 \\
100h + B = 0,
$$

so $h = 1.0488, B = -104.88$, and $F_0 = 110$.

• note that this is consistent with spot-futures parity: $F_0 = S_0(1 + r)^2$ (no-dividends).

• general formulas:

  – to compute futures prices we solve equations of the form:

$$
\begin{align*}
  h_F S^+ + RB &= F^+ - F \\
  h_F S^- + RB &= F^- - F \\
  h_F S + B &= 0
\end{align*}
$$

  – these have the solution:

$$
\begin{align*}
  h_F &= \frac{F^+ - F^-}{S^+ - S^-} = \frac{F^+ - F^-}{S(u - d)} \\
  B &= -h_F S = \frac{F^- - F^+}{(u - d)} \\
  F &= \frac{(R - d)}{(u - d)} \cdot F^+ + \frac{(u - R)}{(u - d)} \cdot F^- = \pi F^+ + (1 - \pi) F^-
\end{align*}
$$

  – this implies that futures prices are the expected value (using the risk-neutral probabilities) of their levels in the future.
we can also replicate the option with the futures contract and the risk free bond. If the stock rises to 114.02 after six months, we have:

\[(130 - 119.58)h_F + 1.0488B = 32\]
\[(100 - 119.58)h_F + 1.0488B = 2.\]

The solution is \(h_F = 1, B = 20.58\). Since it costs nothing to enter the futures contract, the value of the replicating portfolio after six months (and hence \(C_u\)) is just 20.58. Similarly, for \(C_d\):

\[(100 - 91.99)h_F + 1.0488B = 2\]
\[(76.92 - 91.99)h_F + 1.0488B = 0,\]

so \(h_F = .0867, B = 1.24 = C_d\). Finally, working back to \(C_0\):

\[(119.58 - 110)h_F + 1.0488B = 20.58\]
\[(91.99 - 110)h_F + 1.0488B = 1.24,\]

so \(h_F = .7006, B = 13.22 = C_0\).

The Value Line stock index futures contract—a history lesson

- the cost-of-carry relationship for pricing a stock index futures contract at time \(t\) with maturity \(T\) is:

\[F_t = S_t(1 + r - d)^{T-t}\]

where \(d\) is the dividend yield on the index. For example, if the index is 240, \(r = 7\%\), \(d = 4\%\), and \(T - t\) is 3 months, then:

\[F_t = 240(1.03)^{2.5} = 241.78\]

(so the basis is 1.78)

- in the early 1980s, academic research documented the “small firm in January effect”
the Value Line index is a geometrically-averaged index of about 1,700 stocks. It is an equally-weighted index ⇒ relatively intensive in small stocks.

in 1982 futures contracts on the Value Line index and the S&P 500 began trading (expiration months of March, June, September, and December)

one way of capitalizing on the small firm in January effect was to go long the March Value Line index futures contract and short the March S&P 500 contract (hedges against market moves, while profiting if small stocks outperform the market)

strategy worked well in 1983 and 1984, but became too popular by the fall of 1985—so many people were trying it that it became less profitable ⇒ a “calendar spread” strategy: long March 1986 Value Line futures, short December 1985 Value Line futures. In December roll the short position over into a short March S&P futures.

“Everything worked exactly according to plan in December of 1985. My partners and I cleaned up. My futures markets profits for the year exceeded my assistant professor salary … In early January 1986, the Value Line index rose relative to the S&P, and we continued to make money. The basis on the Value Line futures rose too, making us even more.”

“On the afternoon of January 8, 1986, I went off to the classroom to teach. After class, I returned to my office just as the futures market closed. There was a message waiting for me. ‘Your futures broker called. URGENT.’ … I called the futures broker. The first thing he asked was, ‘Are you sitting down?’ While I was teaching, the Value Line index had dropped, and lots of futures market speculators had decided it was time to take their profits and run … We had lost all of our profits. My losses for the afternoon exceeded one-third of my annual salary … I blamed the afternoon’s losses on bad luck. The strategy was sound. I was an informed trader, trading against uninformed traders.”
• in May 1986 Ritter and a partner received news that someone was willing to offer a large position if they wanted to go long the March 1987 Value Line futures and short the December 1986 contract. The partner was skeptical (was this really an uninformed trader?). Ritter took a long position of several million dollars of the March contract and a short position of the same amount for the December contract. The open interest in the newspaper showed that Ritter’s group held the entire long side of the March contract.

• in June 1986 the strategy started to go wrong. The basis on the March contract fell, while it did not on the December contract. Ritter received a series of margin calls. For 5 days in a row, he lost $5,000. What was happening?

Ritter remembered that because the Value Line index was geometric, there was a slight difference in the cost-of-carry relationship. A 1983 paper in the *Journal of Futures Markets* had contained an appendix which showed that for a geometric index:

\[ F_t = S_t (1 + r - d - \frac{1}{2} \bar{s}_t)^{T-t} \]

⇒ since \( \bar{s}_t \approx 10\% \), the correct basis factor was not 1.03 but 0.98

• “I had a multi-million dollar position in a futures contract that the market was pricing using the wrong formula, and now the market was moving towards pricing it using the correct formula . . . I stood to lose another $50,000 if I couldn’t unwind my position. There was only one problem . . . The positions we had were 100% of the long positions in the March contract. The only party on the other side had obviously figured out the correct formula, and this party wasn’t about to let us out at the ‘market price’. We had discovered what an ‘illiquid’ position meant.”
Ritter figured out that he might be able to cut his losses by going long the September contract, so he added another multi-million dollar position. His total was long $6 million of September and March, short $12 million of December. “A little wiggle in the term structure could wipe me out. Isn’t leverage great?”

Ritter did manage to cut his losses this way. “But for the year as a whole, 1986 was a bad year. I lost more in the futures market than I made from my academic salary. And I decided that maybe I wasn’t an informed trader after all, but instead was one of those traders who think that they are informed, when in reality they are providing the profits to the truly informed investors. Years later, I found out who was on the other side of the trades in the summer of 1986. It was Goldman Sachs, with Fischer Black advising the traders, that took me to the cleaners as the market moved from one pricing regime to another.”